COMMUN. MATH. SCI.
Vol. 24, No. 7, pp. 1999–2022

RECOVERY OF A GENERAL ANALYTIC HAMILTONIAN AND RUNNING COSTS IN MEAN FIELD GAMES*

WEINAN WANG†

Abstract. In this paper, we consider an inverse problem for mean field games (MFGs) where both the Hamiltonian and the running cost are unknown. More precisely, we consider an MFG system on the n-dimensional torus $\mathbb{T}^n$ with a general Hamiltonian $H(x,p)$ analytic in the momentum variable p, and a running cost $F(x,m)$ analytic in the measure variable m. The terminal cost is assumed known. We introduce a measurement map $\mathcal{M}_{H,F}$ that sends initial population distributions to the total cost at time zero. Our main result shows that under appropriate admissibility conditions, the measurement map $\mathcal{M}_{H,F}$ uniquely determines both the Hamiltonian H and the running cost F. The proof employs a higher-order linearization technique that decouples the effects of H and F at different orders of linearization, combined with Fourier analysis and high-frequency asymptotics for the linearized equations.

Keywords. Mean field games; inverse problems; Hamiltonian recovery; unique identifiability; higher-order linearization.

AMS subject classifications. 35R30; 49N70; 35K55; 91A16.

1. Introduction

Mean field game (MFG) theory, introduced independently by Caines–Huang–Malhame [1, 2] and Lasry–Lions [10, 11], provides a powerful analytical framework for studying strategic interactions in large populations of rational agents. The theory formulates these interactions through a coupled system of forward-backward partial differential equations: a Hamilton–Jacobi–Bellman equation governing the optimal cost of a representative agent, and a Fokker–Planck equation describing the evolution of the population distribution. In practical scenarios, agents often face uncertainty about the precise structure of their environment—specifically, the Hamiltonian encoding their kinetic costs and the running cost representing interactions with the population. However, outcomes such as the total cost $u(\cdot,0)$ at the game's inception may be observable. This naturally leads to *inverse problems* for MFGs, where one seeks to recover unknown components of the governing equations from measurable data.

Recent years have seen growing interest in such inverse problems, with contributions spanning both numerical [3, 7] and theoretical approaches. On the theoretical side, Liu–Mou–Zhang [25] established several unique identifiability results under various assumptions: when the Hamiltonian is quadratic and the running cost unknown; when the Hamiltonian is known (but general) and the running cost unknown; or when both the running and terminal costs are unknown but the Hamiltonian remains quadratic. Their work left open the question of whether both a *general analytic Hamiltonian* and the running cost could be determined simultaneously from interior measurements. Concurrently, a distinct line of inquiry has explored inverse problems using boundary measurements. Liu and Lo [14] considered the challenging scenario where not only the Hamiltonian and running cost are unknown, but also the system's stationary state. By employing complex geometric optics solutions and linearizing around an unknown nontrivial steady state, they proved simultaneous recovery of a quadratic Hamiltonian, running cost, and the stationary solution from Cauchy data on the boundary. Their

*Received: January 09, 2026; Accepted (in revised form): March 25, 2026. Communicated by Bangti Jin.

†Department of Mathematics, University of Oklahoma, Norman, OK, USA (ww@ou.edu).

arguments rely crucially on the quadratic structure of the Hamiltonian and the availability of complex geometric optics solutions, techniques that do not extend to general analytic Hamiltonians. Related works [15–18] have further developed the boundary measurement approach under various constraints. Meanwhile, other researchers have investigated inverse problems using final-time [9] or internal [8] measurements, often employing Carleman estimates to establish stability. Beyond the specific context of MFGs, recent progress has also been made on inverse problems for other nonlinear evolution equations, including active scalar equations [12] with fractional Laplacian and semilinear wave equations [13] with partial data, as well as for non-local models governed by fractional and integro-differential operators [4, 5, 15, 19–24, 26].

The present paper addresses the aforementioned open question from a different angle. We consider a measurement map $\mathcal{M}_{H,F}$ that sends small initial distributions to the resulting total cost at time zero—an arguably natural observable when one can influence the initial population and measure the ensuing aggregate cost. Within this framework, we prove that both a *general analytic Hamiltonian* $H(x,p)$ and an analytic running cost $F(x,m)$ can be uniquely recovered. Our setting differs substantively from the boundary-measurement paradigm in several key aspects: we consider interior measurements rather than boundary data; our Hamiltonian is not restricted to quadratic forms but belongs to the broad class of analytic functions; and the stationary state in our setting is trivial (the zero state) rather than an additional unknown to be recovered. Methodologically, our approach extends the higher-order linearization technique of [25] but requires novel refinements to disentangle the coupled influences of H and F. At each order of linearization, the Hamiltonian contributes terms involving products of spatial derivatives of the value function, whereas the running cost contributes source terms depending on products of the population distribution. A key technical innovation is the development of rigorous high-frequency asymptotics for solutions to the linearized equations, which allows us to separate these different types of terms through Fourier analysis. The proof combines parabolic well-posedness theory, adjoint methods, and harmonic analysis on the torus.

Thus our work complements the concurrent result of Liu and Lo [14]: while they recover a quadratic Hamiltonian alongside an unknown stationary state from boundary data, we recover a general analytic Hamiltonian from initial-to-final interior measurements. Together, these works address different but equally important scenarios and employ distinct technical approaches, significantly expanding the known scope of simultaneously identifiable parameters in MFG systems.

The rest of the paper is organized as follows. Section 2 introduces notation, the MFG system, admissibility classes, and states our main theorem. Section 3 establishes local well-posedness of the forward problem. Section 4 details the higher-order linearization procedure. Section 5 contains the complete proof of unique identifiability, including the crucial high-frequency asymptotics lemmas.

2. Preliminaries and main results

2.1. Notation and function spaces.

Let $n \in \mathbb{N}$ and $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus. For $\alpha \in (0,1)$ and $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we denote by $C^{k+\alpha}(\mathbb{T}^n)$ the space of functions whose derivatives up to order k are α-Hölder continuous, with norm

$$\|f\|_{C^{k+\alpha}(\mathbb{T}^n)} := \sum_{|\beta| \le k} \sup_{x \in \mathbb{T}^n} |D^\beta f(x)| + \sum_{|\beta|=k} \sup_{x \neq y} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\alpha}.$$

For functions on $\mathbb{T}^n\times[0,T]$, we use the parabolic Hölder spaces $C^{k+\alpha,\frac{k+\alpha}{2}}(\mathbb{T}^n\times[0,T])$ with norm

$$\|f\|_{C^{k+\alpha,\frac{k+\alpha}{2}}}:=\sum_{|\beta|+2j\le k}\sup_Q|D_x^\beta D_t^j f|+\sum_{|\beta|+2j=k}\sup_{(x,t)\ne(y,s)}\frac{|D_x^\beta D_t^j f(x,t)-D_x^\beta D_t^j f(y,s)|}{|x-y|^\alpha+|t-s|^{\alpha/2}}.$$

Define $Q:=\mathbb{T}^n\times[0,T]$ and let $\mathcal{P}(\mathbb{T}^n)$ denote the space of probability measures on $\mathbb{T}^n$. We consider initial distributions in the set

$$\mathcal{O}_a:=\left\{m_0\in C^{2+\alpha}(\mathbb{T}^n):m_0\ge 0,\int_{\mathbb{T}^n}m_0(x)\,dx=a\right\},$$

where $a\in[0,1]$. As explained in [25], this allows for the possibility that the game domain consists of several disjoint subdomains.

2.2. Admissibility classes. We introduce analyticity classes for the Hamiltonian H and running cost F.

DEFINITION 2.1 (Admissible Hamiltonian). *A function $H:\mathbb{T}^n\times\mathbb{C}^n\to\mathbb{C}$ is called* admissible, *denoted $H\in\mathcal{H}$, if:*

(1) *For each fixed $p\in\mathbb{C}^n$, $H(\cdot,p)\in C^{2+\alpha}(\mathbb{T}^n)$, and the map $p\mapsto H(\cdot,p)$ is holomorphic from $\mathbb{C}^n$ to $C^{2+\alpha}(\mathbb{T}^n)$ for some $\alpha\in(0,1)$;*

(2) *$H(x,0)=0$ for all $x\in\mathbb{T}^n$.*

For $H\in\mathcal{H}$, we have the power series expansion

$$H(x,p)=\sum_{k=1}^{\infty}\sum_{|\beta|=k}H^{(\beta)}(x)\frac{p^\beta}{\beta!},$$

where $\beta=(\beta_1,\dots,\beta_n)\in\mathbb{N}_0^n$ is a multi-index, $p^\beta=p_1^{\beta_1}\cdots p_n^{\beta_n}$, $\beta!=\beta_1!\cdots\beta_n!$, and $H^{(\beta)}\in C^{2+\alpha}(\mathbb{T}^n)$. The series converges absolutely and uniformly on compact subsets of $\mathbb{C}^n$. We assume all coefficient functions are real-valued when restricted to real arguments.

REMARK 2.1 (Non-quadratic Hamiltonians and the adjoint structure). The class $\mathcal{H}$ includes all analytic Hamiltonians vanishing at $p=0$, in particular non-quadratic ones. As a concrete example, for $p\in\mathbb{R}^n$ the cubic ($|\beta|=3$) Hamiltonian takes the form

$$H(x,p)=\sum_i a_i(x)p_i+\frac{1}{2}\sum_{i,j}b_{ij}(x)p_ip_j+\frac{1}{6}\sum_{i,j,k}c_{ijk}(x)p_ip_jp_k,$$

with $a_i,b_{ij},c_{ijk}\in C^{2+\alpha}(\mathbb{T}^n)$ admissible in the sense of Definition 2.1. Here $H^{(e_i)}(x)=a_i(x)$, $H^{(e_i+e_j)}(x)=b_{ij}(x)$, and $H^{(e_i+e_j+e_k)}(x)=c_{ijk}(x)$. A representative purely cubic choice is $H(x,p)=\sigma(x)(p_1^3+p_1p_2^2+\cdots+p_n^3)$ for $\sigma\in C^{2+\alpha}(\mathbb{T}^n)$, arising in models with super-linear velocity costs.

We also explain why the MFG system (2.3) is a pair of adjoint equations. Setting $A^{(1)}(x)=H_p(x,0)$, the first-order linearised system (cf. Section 4) reads

$$\begin{cases}-\partial_t u^{(1)}-\Delta u^{(1)}+A^{(1)}(x)\cdot\nabla u^{(1)}=F^{(1)}(x)m^{(1)},\\ \partial_t m^{(1)}-\Delta m^{(1)}-\operatorname{div}(m^{(1)}A^{(1)}(x))=0.\end{cases}$$

Defining $\mathcal{L}\phi:=-\partial_t\phi-\Delta\phi+A^{(1)}\cdot\nabla\phi$, its formal L^2-adjoint is $\mathcal{L}^*\psi=\partial_t\psi-\Delta\psi-\operatorname{div}(A^{(1)}\psi)$, which is precisely the operator governing $m^{(1)}$. This adjoint pairing enables the integration-by-parts identities in Lemma 5.2 and is a structural feature of all

MFG systems derived from optimal control: the HJB equation and the Fokker–Planck equation are always adjoints with respect to the optimal drift, regardless of whether H is quadratic.

DEFINITION 2.2 (Admissible running cost). *A function $F:\mathbb{T}^n\times\mathbb{C}\to\mathbb{C}$ is called* admissible, *denoted $F\in\mathcal{F}$, if:*

(1) *For each fixed $z\in\mathbb{C}$, $F(\cdot,z)\in C^{2+\alpha}(\mathbb{T}^n)$, and the map $z\mapsto F(\cdot,z)$ is holomorphic from $\mathbb{C}$ to $C^{2+\alpha}(\mathbb{T}^n)$ for some $\alpha\in(0,1)$;*

(2) *$F(x,0)=0$ for all $x\in\mathbb{T}^n$.*

For $F\in\mathcal{F}$, we have the expansion

$$F(x,z)=\sum_{k=1}^{\infty}F^{(k)}(x)\frac{z^k}{k!},$$

with $F^{(k)}\in C^{2+\alpha}(\mathbb{T}^n)$, converging absolutely and uniformly on compact subsets of $\mathbb{C}$.

2.3. Forward MFG system. We consider the MFG system:

$$\begin{cases}-\partial_t u(x,t)-\Delta u(x,t)+H(x,\nabla u(x,t))=F(x,m(x,t)), & \text{in } \mathbb{T}^n\times(0,T),\\ \partial_t m(x,t)-\Delta m(x,t)-\operatorname{div}\left(m(x,t)H_p(x,\nabla u(x,t))\right)=0, & \text{in } \mathbb{T}^n\times(0,T),\\ u(x,T)=G(x,m(x,T)),\quad m(x,0)=m_0(x), & \text{in } \mathbb{T}^n,\end{cases}\tag{2.1}$$

where $G:\mathbb{T}^n\times\mathbb{R}\to\mathbb{R}$ is a known terminal cost. We assume G satisfies similar analyticity conditions: $G\in\mathcal{G}$, where $\mathcal{G}$ is defined analogously to $\mathcal{F}$ with $G(x,0)=0$.

2.4. Inverse problem and measurement map. For given $H\in\mathcal{H}$ and $F\in\mathcal{F}$, let (u,m) be the solution to (2.3) with initial condition $m_0\in\mathcal{O}_a$. Define the *measurement map*

$$\mathcal{M}_{H,F}:\mathcal{O}_a\to C^{2+\alpha}(\mathbb{T}^n),\quad \mathcal{M}_{H,F}(m_0):=u(\cdot,0).$$

This map sends an initial population distribution to the total cost at time zero. The inverse problem we study is:

$$\text{Given } \mathcal{M}_{H,F}, \text{ determine } H \text{ and } F.$$

2.5. Main result. Our main theorem establishes the unique identifiability of both H and F from the measurement map.

THEOREM 2.1 (Simultaneous recovery of H and F). *Let $H_1,H_2\in\mathcal{H}$ and $F_1,F_2\in\mathcal{F}$. Let $G\in\mathcal{G}$ be a known terminal cost. For $j=1,2$, let $\mathcal{M}_{H_j,F_j}$ be the measurement map associated with the MFG system (2.3) with Hamiltonian H_j and running cost F_j. Suppose there exists $\delta>0$ such that for all $m_0\in\mathcal{O}_a$ with $\|m_0\|_{C^{2+\alpha}(\mathbb{T}^n)}\le\delta$, we have*

$$\mathcal{M}_{H_1,F_1}(m_0)=\mathcal{M}_{H_2,F_2}(m_0).$$

Then

$$H_1(x,p)=H_2(x,p)\quad \textit{and}\quad F_1(x,z)=F_2(x,z)$$

for all $x\in\mathbb{T}^n$, $p\in\mathbb{R}^n$, and $z\in\mathbb{R}$.

REMARK 2.2. The requirement that m_0 be small is needed for the well-posedness of the forward problem (Theorem 3.1). The equality of measurement maps for small m_0

suffices to determine the Taylor coefficients of H and F at the origin, which uniquely determine the analytic functions.

Remark 2.3. Unlike Theorem 6.1 in [25] which assumes H is known, our theorem allows both H and F to be unknown. The proof requires analyzing the coupling between their Taylor coefficients at each order of linearization.

3. Well-posedness of the forward problem

In this section, we establish the local well-posedness of the MFG system (2.3) for small initial data, which is needed for the higher-order linearization procedure.

Lemma 3.1 (Linear parabolic estimates). *Let $A\in C^{\alpha,\alpha/2}(Q;\mathbb{R}^n)$, $f\in C^{\alpha,\alpha/2}(Q)$, and $v_0\in C^{2+\alpha}(\mathbb{T}^n)$. Consider the initial value problem*

$$\begin{cases}\partial_t v-\Delta v-\operatorname{div}(A(x,t)v)=f(x,t), & \text{in } \mathbb{T}^n\times(0,T),\\ v(x,0)=v_0(x), & \text{in } \mathbb{T}^n.\end{cases}$$

There exists a unique solution $v\in C^{2+\alpha,1+\alpha/2}(Q)$ satisfying

$$\|v\|_{C^{2+\alpha,1+\alpha/2}(Q)}\le C(\|v_0\|_{C^{2+\alpha}(\mathbb{T}^n)}+\|f\|_{C^{\alpha,\alpha/2}(Q)}),$$

where C depends on $\|A\|_{C^{\alpha,\alpha/2}(Q)}$, T, and n.

Proof. We rewrite the equation as

$$\partial_t v-\Delta v-A\cdot\nabla v-(\operatorname{div} A)v=f.$$

Since $A\in C^{\alpha,\alpha/2}(Q;\mathbb{R}^n)$, we have $\operatorname{div} A\in C^{\alpha-1,\frac{\alpha-1}{2}}(Q)$ if $\alpha>1$. Define the operator $\mathcal{L}v=\partial_t v-\Delta v-A\cdot\nabla v$. First, we establish an a priori estimate. Multiply the equation by v and integrate:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2+\|\nabla v\|_{L^2}^2&=\int_{\mathbb{T}^n}(A\cdot\nabla v+(\operatorname{div} A)v+f)v\,dx\\ &\le\|A\|_{L^\infty}\|\nabla v\|_{L^2}\|v\|_{L^2}+\|\operatorname{div} A\|_{L^\infty}\|v\|_{L^2}^2+\|f\|_{L^2}\|v\|_{L^2}.\end{aligned}$$

Using Young's inequality and Grönwall's lemma yields $\|v\|_{L^\infty(0,T;L^2)}+\|\nabla v\|_{L^2(Q)}\le C$. For Hölder estimates, we use the method of freezing coefficients. Fix $(x_0,t_0)\in Q$ and consider the constant coefficient operator

$$\mathcal{L}_0 v=\partial_t v-\Delta v-A(x_0,t_0)\cdot\nabla v.$$

The fundamental solution $\Gamma_0(x,t;y,s)$ for $\mathcal{L}_0$ satisfies

$$|\partial_x^\beta\partial_y^\gamma\Gamma_0(x,t;y,s)|\le C(t-s)^{-(n+|\beta|+|\gamma|)/2}e^{-c|x-y|^2/(t-s)}.$$

Write $v=v_1+v_2$ where v_1 solves $\mathcal{L}_0 v_1=f$ with zero initial data, and v_2 solves $\mathcal{L}_0 v_2=(A(x,t)-A(x_0,t_0))\cdot\nabla v+(\operatorname{div} A)v$ with initial data v_0. By the Schauder theory for constant coefficient operators (Theorem 5.1 in [27]),

$$\|v_1\|_{C^{2+\alpha,1+\alpha/2}}\le C\|f\|_{C^{\alpha,\alpha/2}}.$$

For v_2, we use a fixed point argument. Define the mapping $\mathcal{T}:w\mapsto z$ where z solves

$$\mathcal{L}_0 z=(A(x,t)-A(x_0,t_0))\cdot\nabla w+(\operatorname{div} A)w,\quad z(x,0)=v_0(x).$$

Since $|A(x,t)-A(x_0,t_0)|\le C(|x-x_0|^\alpha+|t-t_0|^{\alpha/2})$, the operator $\mathcal{T}$ is a contraction on $C^{1+\alpha,\frac{1+\alpha}{2}}$ for sufficiently small T or with appropriate weights. The existence of a unique fixed point follows from Banach's fixed point theorem. Uniqueness follows from the maximum principle. More precisely, if v_1,v_2 are two solutions, their difference w satisfies $\mathcal{L}w=0$ with $w(x,0)=0$. Multiplying by w and integrating gives $\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2+\|\nabla w\|_{L^2}^2\le C\|w\|_{L^2}^2$, so by Grönwall, $w\equiv 0$. ∎

Now we prove the well-posedness of the nonlinear MFG system.

THEOREM 3.1 (Well-posedness). *Let $H\in\mathcal{H}$, $F\in\mathcal{F}$, and $G\in\mathcal{G}$. There exist constants $\delta>0$ and $C>0$ such that for any $m_0\in C^{2+\alpha}(\mathbb{T}^n)$ with $\|m_0\|_{C^{2+\alpha}(\mathbb{T}^n)}\le\delta$, the MFG system (2.3) has a unique solution $(u,m)\in C^{2+\alpha,1+\frac{\alpha}{2}}(Q)\times C^{2+\alpha,1+\frac{\alpha}{2}}(Q)$ satisfying*

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q)}+\|m\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q)}\le C\|m_0\|_{C^{2+\alpha}(\mathbb{T}^n)}.$$

Moreover, the solution map $S:m_0\mapsto(u,m)$ is holomorphic from $B_\delta(C^{2+\alpha}(\mathbb{T}^n))$ to $C^{2+\alpha,1+\frac{\alpha}{2}}(Q)^2$.

Proof. We use the implicit function theorem in Banach spaces. Define:

$$\begin{aligned}
X_1&:=C^{2+\alpha}(\mathbb{T}^n),\\
X_2&:=C^{2+\alpha,1+\frac{\alpha}{2}}(Q)\times C^{2+\alpha,1+\frac{\alpha}{2}}(Q),\\
X_3&:=C^{2+\alpha}(\mathbb{T}^n)\times C^{2+\alpha}(\mathbb{T}^n)\times C^{\alpha,\frac{\alpha}{2}}(Q)\times C^{\alpha,\frac{\alpha}{2}}(Q).
\end{aligned}$$

Define $\Phi:X_1\times X_2\to X_3$ by

$$\begin{aligned}
\Phi(m_0,u,m):=\big(&u(\cdot,T)-G(\cdot,m(\cdot,T)),\\
&m(\cdot,0)-m_0,\\
&-\partial_t u-\Delta u+H(\cdot,\nabla u)-F(\cdot,m),\\
&\partial_t m-\Delta m-\operatorname{div}(mH_p(\cdot,\nabla u))\big).
\end{aligned}$$

Φ is well-defined and holomorphic. Since H is holomorphic in p, $H(\cdot,\nabla u)$ is a holomorphic function of ∇u with values in $C^{\alpha,\alpha/2}(Q)$. Similarly, $F(\cdot,m)$ and $G(\cdot,m)$ are holomorphic in m. The derivative terms preserve the regularity. Thus Φ is holomorphic (see [28, Theorem 1]). We see $\Phi(0,0,0)=0$, which follows from $H(x,0)=0$, $F(x,0)=0$, and $G(x,0)=0$. For the derivative at $(0,0,0)$, we compute $D_{(u,m)}\Phi(0,0,0)(v,\mu)$:

$$\begin{aligned}
D_{(u,m)}\Phi_1(0,0,0)(v,\mu)&=v(\cdot,T)-G^{(1)}(\cdot)\mu(\cdot,T),\\
D_{(u,m)}\Phi_2(0,0,0)(v,\mu)&=\mu(\cdot,0),\\
D_{(u,m)}\Phi_3(0,0,0)(v,\mu)&=-\partial_t v-\Delta v+H_p(\cdot,0)\cdot\nabla v-F^{(1)}(\cdot)\mu,\\
D_{(u,m)}\Phi_4(0,0,0)(v,\mu)&=\partial_t\mu-\Delta\mu-\operatorname{div}(\mu H_p(\cdot,0)).
\end{aligned}$$

Let $A^{(1)}(x):=H_p(x,0)\in C^{2+\alpha}(\mathbb{T}^n;\mathbb{R}^n)$. To show $D_{(u,m)}\Phi(0,0,0)$ is an isomorphism, we need to show that for any $(r_1,r_2,f_1,f_2)\in X_3$, there exists a unique $(v,\mu)\in X_2$ such that

$$v(\cdot,T)-G^{(1)}(\cdot)\mu(\cdot,T)=r_1,$$
$$\mu(\cdot,0)=r_2, \tag{3.1}$$
$$-\partial_t v-\Delta v+A^{(1)}\cdot\nabla v-F^{(1)}\mu=f_1, \tag{3.2}$$
$$\partial_t\mu-\Delta\mu-\operatorname{div}(\mu A^{(1)})=f_2. \tag{3.3}$$

First, solve (3) with initial condition (3). This is a linear parabolic equation of the form

$$\partial_t \mu - \Delta \mu - A^{(1)} \cdot \nabla \mu - (\operatorname{div} A^{(1)})\mu = f_2.$$

By Lemma 3.1 (applied with drift $A^{(1)}$ and right-hand side f_2), there exists a unique solution $\mu \in C^{2+\alpha,1+\alpha/2}(Q)$ with

$$\|\mu\|_{C^{2+\alpha,1+\alpha/2}} \le C(\|r_2\|_{C^{2+\alpha}} + \|f_2\|_{C^{\alpha,\alpha/2}}).$$

Now solve (3) backward in time. Let $\tau = T - t$, $\tilde{v}(x,\tau) = v(x,T-\tau)$, $\tilde{\mu}(x,\tau) = \mu(x,T-\tau)$, $\tilde{f}_1(x,\tau) = f_1(x,T-\tau)$. Then

$$\partial_\tau \tilde{v} - \Delta \tilde{v} + A^{(1)} \cdot \nabla \tilde{v} = F^{(1)}\tilde{\mu} + \tilde{f}_1,$$

with initial condition $\tilde{v}(x,0) = G^{(1)}(x)\mu(x,T) + r_1(x)$. Again by Lemma 3.1, there exists a unique solution $\tilde{v} \in C^{2+\alpha,1+\alpha/2}(Q)$, hence $v \in C^{2+\alpha,1+\alpha/2}(Q)$. To show the linear map $(r_1,r_2,f_1,f_2) \mapsto (v,\mu)$ is bounded, we need to estimate $\|v\|_{C^{2+\alpha,1+\alpha/2}}$. From the solution of (3):

$$\|\mu\|_{C^{2+\alpha,1+\alpha/2}} \le C_1(\|r_2\|_{C^{2+\alpha}} + \|f_2\|_{C^{\alpha,\alpha/2}}).$$

For v, we have two contributions: the homogeneous solution with initial data $G^{(1)}\mu(\cdot,T) + r_1$, and the particular solution from $F^{(1)}\mu + f_1$. By linearity and Lemma 3.1:

$$\begin{aligned}\|v\|_{C^{2+\alpha,1+\alpha/2}} &\le C_2(\|G^{(1)}\mu(\cdot,T) + r_1\|_{C^{2+\alpha}} + \|F^{(1)}\mu + f_1\|_{C^{\alpha,\alpha/2}}) \\ &\le C_3(\|\mu\|_{C^{2+\alpha,1+\alpha/2}} + \|r_1\|_{C^{2+\alpha}} + \|f_1\|_{C^{\alpha,\alpha/2}}) \\ &\le C_4(\|r_1\|_{C^{2+\alpha}} + \|r_2\|_{C^{2+\alpha}} + \|f_1\|_{C^{\alpha,\alpha/2}} + \|f_2\|_{C^{\alpha,\alpha/2}}).\end{aligned}$$

Thus $D_{(u,m)}\Phi(0,0,0)$ is bounded. Injectivity follows from uniqueness of solutions to linear parabolic equations. Surjectivity follows from our construction. By the open mapping theorem, it is an isomorphism. By the analytic implicit function theorem in Banach spaces (Theorem 15.3 in [6]), there exist neighborhoods $U \subset X_1$ of 0 and $V \subset X_2$ of $(0,0)$, and a holomorphic map $S : U \to V$ such that $\Phi(m_0, S(m_0)) = 0$ for all $m_0 \in U$, and $S(0) = (0,0)$. To ensure S maps into probability measures (nonnegative with integral a), we need to verify that if $m_0 \in \mathcal{O}_a$, then $m(\cdot,t) \ge 0$ and $\int_{\mathbb{T}^n} m(x,t)dx = a$ for all t. This follows from the maximum principle for the Fokker-Planck equation and conservation of mass:

$$\frac{d}{dt}\int_{\mathbb{T}^n} m(x,t)dx = \int_{\mathbb{T}^n} [\Delta m + \operatorname{div}(mH_p(x,\nabla u))]dx = 0.$$

Thus $\int_{\mathbb{T}^n} m(x,t)dx = \int_{\mathbb{T}^n} m_0(x)dx = a$. Next, we show uniqueness. If (u_1,m_1) and (u_2,m_2) are two solutions in V, then $\Phi(m_0,u_1,m_1) = \Phi(m_0,u_2,m_2) = 0$. Since S is uniquely defined, we have $(u_1,m_1) = (u_2,m_2)$. ∎

Corollary 3.1. *The measurement map $\mathcal{M}_{H,F} : m_0 \mapsto u(\cdot,0)$ is holomorphic on $B_\delta(C^{2+\alpha}(\mathbb{T}^n))$.*

Proof. $\mathcal{M}_{H,F} = \pi_1 \circ S|_{t=0}$, where $\pi_1 : X_2 \to C^{2+\alpha,1+\alpha/2}(Q)$ is projection onto the first component and S is the solution map from Theorem 3.1. The evaluation at $t=0$ is a bounded linear operator from $C^{2+\alpha,1+\alpha/2}(Q)$ to $C^{2+\alpha}(\mathbb{T}^n)$. The composition of holomorphic maps is holomorphic. ∎

4. Higher-order linearization

The key tool for proving Theorem 2.1 is a higher-order linearization technique. We perturb the initial condition m_0 by multiple parameters and differentiate the resulting solution with respect to these parameters.

4.1. Notation. Let $N\in\mathbb{N}$ and $f_1,\dots,f_N\in C_+^{2+\alpha}(\mathbb{T}^n)$ (nonnegative functions). For $\varepsilon=(\varepsilon_1,\dots,\varepsilon_N)\in[0,\delta)^N$ with δ from Theorem 3.1, define

$$m_0(x;\varepsilon):=\sum_{l=1}^N \varepsilon_l f_l(x).$$

By Theorem 3.1, for $|\varepsilon|$ sufficiently small, there exists a unique solution $(u(x,t;\varepsilon),m(x,t;\varepsilon))$ to (2.3) with initial condition $m_0(\cdot;\varepsilon)$. Note that $(u(x,t;0),m(x,t;0))=(0,0)$ due to the zero conditions on H, F, and G. For a multi-index $\alpha=(\alpha_1,\dots,\alpha_N)\in\mathbb{N}_0^N$, we define

$$u^{(\alpha)}(x,t):=\frac{\partial^{|\alpha|}u}{\partial\varepsilon^\alpha}(x,t;0),\quad m^{(\alpha)}(x,t):=\frac{\partial^{|\alpha|}m}{\partial\varepsilon^\alpha}(x,t;0),$$

where $|\alpha|=\alpha_1+\cdots+\alpha_N$ and $\partial\varepsilon^\alpha=\partial\varepsilon_1^{\alpha_1}\cdots\partial\varepsilon_N^{\alpha_N}$. We will particularly need derivatives with respect to single parameters and pairs of parameters. For $l\in\{1,\dots,N\}$, denote

$$u^{(l)}(x,t):=\frac{\partial u}{\partial\varepsilon_l}(x,t;0),\quad m^{(l)}(x,t):=\frac{\partial m}{\partial\varepsilon_l}(x,t;0).$$

For $l,k\in\{1,\dots,N\}$, denote

$$u^{(l,k)}(x,t):=\frac{\partial^2 u}{\partial\varepsilon_l\partial\varepsilon_k}(x,t;0),\quad m^{(l,k)}(x,t):=\frac{\partial^2 m}{\partial\varepsilon_l\partial\varepsilon_k}(x,t;0).$$

4.2. First-order linearization. We differentiate (2.3) with respect to ε_l at $\varepsilon=0$.

LEMMA 4.1 (First-order linearized system). *The pair $(u^{(l)},m^{(l)})$ satisfies*

$$\begin{cases}-\partial_t u^{(l)}-\Delta u^{(l)}+A^{(1)}(x)\cdot\nabla u^{(l)}=F^{(1)}(x)m^{(l)}, & \text{in } \mathbb{T}^n\times(0,T),\\ \partial_t m^{(l)}-\Delta m^{(l)}-\operatorname{div}\left(m^{(l)}A^{(1)}(x)\right)=0, & \text{in } \mathbb{T}^n\times(0,T),\\ u^{(l)}(x,T)=G^{(1)}(x)m^{(l)}(x,T), & \text{in } \mathbb{T}^n,\\ m^{(l)}(x,0)=f_l(x), & \text{in } \mathbb{T}^n,\end{cases}\tag{4.1}$$

where $A^{(1)}(x)=H_p(x,0)=(H^{(e_1)}(x),\dots,H^{(e_n)}(x))$ with e_i the standard basis vectors.

Proof. We differentiate each equation in (2.3) term by term at $\varepsilon=0$.

(1) Hamilton–Jacobi equation:

$$\begin{aligned}&\frac{\partial}{\partial\varepsilon_l}\left[-\partial_t u-\Delta u+H(x,\nabla u)-F(x,m)\right]\Big|_{\varepsilon=0}\\ &=-\partial_t u^{(l)}-\Delta u^{(l)}+\frac{\partial}{\partial\varepsilon_l}H(x,\nabla u)\Big|_{\varepsilon=0}-\frac{\partial}{\partial\varepsilon_l}F(x,m)\Big|_{\varepsilon=0}.\end{aligned}$$

For H, using the chain rule and $H(x,0)=0$:

$$\frac{\partial}{\partial\varepsilon_l}H(x,\nabla u)\Big|_{\varepsilon=0}=H_p(x,\nabla u)\Big|_{\varepsilon=0}\cdot\nabla u^{(l)}=H_p(x,0)\cdot\nabla u^{(l)}=A^{(1)}(x)\cdot\nabla u^{(l)}.$$

For F, using $F(x,0)=0$:

$$\frac{\partial}{\partial \varepsilon_l} F(x,m)\big|_{\varepsilon=0} = F_z(x,m)\big|_{\varepsilon=0} m^{(l)} = F^{(1)}(x) m^{(l)}.$$

Thus

$$-\partial_t u^{(l)} - \Delta u^{(l)} + A^{(1)}(x)\cdot \nabla u^{(l)} - F^{(1)}(x) m^{(l)} = 0.$$

(2) Fokker–Planck equation:

$$\begin{aligned}
&\frac{\partial}{\partial \varepsilon_l}\big[\partial_t m - \Delta m - \operatorname{div}(m H_p(x,\nabla u))\big]\big|_{\varepsilon=0} \\
=&\partial_t m^{(l)} - \Delta m^{(l)} - \frac{\partial}{\partial \varepsilon_l}\big[\operatorname{div}(m H_p(x,\nabla u))\big]\big|_{\varepsilon=0}.
\end{aligned}$$

For the divergence term:

$$\begin{aligned}
\frac{\partial}{\partial \varepsilon_l}\big[\operatorname{div}(m H_p(x,\nabla u))\big]\big|_{\varepsilon=0} &= \operatorname{div}\big(m^{(l)} H_p(x,\nabla u)\big|_{\varepsilon=0}\big) + \operatorname{div}\big(m\big|_{\varepsilon=0} \frac{\partial}{\partial \varepsilon_l} H_p(x,\nabla u)\big|_{\varepsilon=0}\big) \\
&= \operatorname{div}\big(m^{(l)} H_p(x,0)\big) + \operatorname{div}\big(0\cdot H_{pp}(x,0)\nabla u^{(l)}\big) \\
&= \operatorname{div}\big(m^{(l)} A^{(1)}(x)\big).
\end{aligned}$$

Thus

$$\partial_t m^{(l)} - \Delta m^{(l)} - \operatorname{div}\big(m^{(l)} A^{(1)}(x)\big) = 0.$$

(3) Terminal condition:

$$\frac{\partial}{\partial \varepsilon_l}\big[u(x,T) - G(x,m(x,T))\big]\big|_{\varepsilon=0} = u^{(l)}(x,T) - G^{(1)}(x) m^{(l)}(x,T) = 0.$$

(4) Initial condition:

$$\frac{\partial}{\partial \varepsilon_l}\big[m(x,0) - m_0(x;\varepsilon)\big]\big|_{\varepsilon=0} = m^{(l)}(x,0) - f_l(x) = 0.$$

□

4.3. Second-order linearization. Now differentiate twice. For $l,k\in\{1,\dots,N\}$, we compute $u^{(l,k)}$ and $m^{(l,k)}$.

LEMMA 4.2 (Second-order linearized system). *The pair $(u^{(l,k)}, m^{(l,k)})$ satisfies*

$$\begin{cases}
\begin{aligned}&-\partial_t u^{(l,k)} - \Delta u^{(l,k)} + A^{(1)}(x)\cdot\nabla u^{(l,k)} + \sum_{|\beta|=2} H^{(\beta)}(x)\partial^\beta (u^{(l)},u^{(k)}) \\ &\quad = F^{(1)}(x) m^{(l,k)} + F^{(2)}(x) m^{(l)} m^{(k)},\end{aligned} & \text{in } \mathbb{T}^n\times(0,T), \\
\partial_t m^{(l,k)} - \Delta m^{(l,k)} - \operatorname{div}\big(m^{(l,k)} A^{(1)}(x)\big) = R^{(l,k)}, & \text{in } \mathbb{T}^n\times(0,T), \\
u^{(l,k)}(x,T) = G^{(1)}(x) m^{(l,k)}(x,T) + G^{(2)}(x) m^{(l)}(x,T) m^{(k)}(x,T), & \text{in } \mathbb{T}^n, \\
m^{(l,k)}(x,0) = 0, & \text{in } \mathbb{T}^n,
\end{cases}$$

where

$$\partial^\beta(u^{(l)},u^{(k)})=\frac{1}{\beta!}\sum_{\sigma\in S_2}\prod_{i=1}^n\partial_{x_i}^{\beta_i}u^{(\sigma(l))}\partial_{x_i}^{\beta_i}u^{(\sigma(k))}$$

for $|\beta|=2$ *(sum over the two permutations of* $\{l,k\}$*), and*

$$R^{(l,k)}=\operatorname{div}\left(m^{(l)}B^{(1)}(x)\nabla u^{(k)}\right)+\operatorname{div}\left(m^{(k)}B^{(1)}(x)\nabla u^{(l)}\right),$$

with $B^{(1)}(x)=H_{pp}(x,0)$ *the Hessian matrix of* H *in* p *at* $p=0$*.*

Proof. We differentiate each equation twice.

(1) Hamilton–Jacobi equation, second derivative:

$$\begin{aligned}&\frac{\partial^2}{\partial\varepsilon_l\partial\varepsilon_k}\left[-\partial_t u-\Delta u+H(x,\nabla u)-F(x,m)\right]\big|_{\varepsilon=0}\\&=-\partial_t u^{(l,k)}-\Delta u^{(l,k)}+\frac{\partial^2}{\partial\varepsilon_l\partial\varepsilon_k}H(x,\nabla u)\big|_{\varepsilon=0}-\frac{\partial^2}{\partial\varepsilon_l\partial\varepsilon_k}F(x,m)\big|_{\varepsilon=0}.\end{aligned}$$

For $H(x,\nabla u)$, use the multivariate Faà di Bruno formula. Write $p=\nabla u$. Then

$$\begin{aligned}\frac{\partial^2}{\partial\varepsilon_l\partial\varepsilon_k}H(x,\nabla u)\big|_{\varepsilon=0}&=H_p(x,0)\cdot\nabla u^{(l,k)}+(\nabla u^{(l)})^T H_{pp}(x,0)(\nabla u^{(k)})\\&=A^{(1)}(x)\cdot\nabla u^{(l,k)}+\sum_{i,j=1}^n\frac{\partial^2 H}{\partial p_i\partial p_j}(x,0)\partial_{x_i}u^{(l)}\partial_{x_j}u^{(k)}.\end{aligned}$$

The second term can be written as $\sum_{|\beta|=2}H^{(\beta)}(x)\partial^\beta(u^{(l)},u^{(k)})$ where for $\beta=e_i+e_j$:

$$H^{(\beta)}(x)=\frac{\partial^2 H}{\partial p_i\partial p_j}(x,0),\quad \partial^\beta(u^{(l)},u^{(k)})=\partial_{x_i}u^{(l)}\partial_{x_j}u^{(k)}.$$

For $i=j$, $\beta=2e_i$, $H^{(\beta)}(x)=\frac{1}{2}\frac{\partial^2 H}{\partial p_i^2}(x,0)$, and $\partial^\beta(u^{(l)},u^{(k)})=2\partial_{x_i}u^{(l)}\partial_{x_i}u^{(k)}$. For $F(x,m)$, using $F(x,0)=0$ and the chain rule:

$$\frac{\partial^2}{\partial\varepsilon_l\partial\varepsilon_k}F(x,m)\big|_{\varepsilon=0}=F^{(1)}(x)m^{(l,k)}+F^{(2)}(x)m^{(l)}m^{(k)}.$$

Thus the Hamilton–Jacobi part gives:

$$\begin{aligned}&-\partial_t u^{(l,k)}-\Delta u^{(l,k)}+A^{(1)}(x)\cdot\nabla u^{(l,k)}+\sum_{|\beta|=2}H^{(\beta)}(x)\partial^\beta(u^{(l)},u^{(k)})\\&F^{(1)}(x)m^{(l,k)}+F^{(2)}(x)m^{(l)}m^{(k)}.\end{aligned}$$

(2) Fokker–Planck equation, second derivative:

$$\begin{aligned}&\frac{\partial^2}{\partial\varepsilon_l\partial\varepsilon_k}\left[\partial_t m-\Delta m-\operatorname{div}(mH_p(x,\nabla u))\right]\big|_{\varepsilon=0}\\&=\partial_t m^{(l,k)}-\Delta m^{(l,k)}-\frac{\partial^2}{\partial\varepsilon_l\partial\varepsilon_k}\left[\operatorname{div}(mH_p(x,\nabla u))\right]\big|_{\varepsilon=0}.\end{aligned}$$

Compute the divergence term:

$$\begin{aligned}
&\frac{\partial^2}{\partial\varepsilon_l\partial\varepsilon_k}\big[\operatorname{div}(mH_p(x,\nabla u))\big]\big|_{\varepsilon=0}\\
=&\operatorname{div}\big(m^{(l,k)}H_p(x,0)\big)+\operatorname{div}\big(m^{(l)}\frac{\partial}{\partial\varepsilon_k}H_p(x,\nabla u)\big|_{\varepsilon=0}\big)\\
&+\operatorname{div}\big(m^{(k)}\frac{\partial}{\partial\varepsilon_l}H_p(x,\nabla u)\big|_{\varepsilon=0}\big)+\operatorname{div}\big(m\big|_{\varepsilon=0}\frac{\partial^2}{\partial\varepsilon_l\partial\varepsilon_k}H_p(x,\nabla u)\big|_{\varepsilon=0}\big).
\end{aligned}$$

Since $m|_{\varepsilon=0}=0$, the last term vanishes. For the middle terms:

$$\frac{\partial}{\partial\varepsilon_k}H_p(x,\nabla u)\big|_{\varepsilon=0}=H_{pp}(x,0)\nabla u^{(k)}=B^{(1)}(x)\nabla u^{(k)}.$$

Similarly for $\frac{\partial}{\partial\varepsilon_l}H_p(x,\nabla u)$. Thus,

$$\begin{aligned}
&\frac{\partial^2}{\partial\varepsilon_l\partial\varepsilon_k}\big[\operatorname{div}(mH_p(x,\nabla u))\big]\big|_{\varepsilon=0}\\
=&\operatorname{div}\big(m^{(l,k)}A^{(1)}(x)\big)+\operatorname{div}\big(m^{(l)}B^{(1)}(x)\nabla u^{(k)}\big)+\operatorname{div}\big(m^{(k)}B^{(1)}(x)\nabla u^{(l)}\big).
\end{aligned}$$

Therefore,

$$\partial_t m^{(l,k)}-\Delta m^{(l,k)}-\operatorname{div}\big(m^{(l,k)}A^{(1)}(x)\big)=R^{(l,k)}.$$

(3) Terminal condition:

$$\begin{aligned}
&\frac{\partial^2}{\partial\varepsilon_l\partial\varepsilon_k}\big[u(x,T)-G(x,m(x,T))\big]\big|_{\varepsilon=0}\\
=&u^{(l,k)}(x,T)-\big[G^{(1)}(x)m^{(l,k)}(x,T)+G^{(2)}(x)m^{(l)}(x,T)m^{(k)}(x,T)\big]=0.
\end{aligned}$$

(4) Initial condition: Since $m_0(x;\varepsilon)=\sum_{j=1}^N\varepsilon_j f_j(x)$ is linear in ε, its second derivatives vanish. Thus, we have

$$m^{(l,k)}(x,0)=\frac{\partial^2}{\partial\varepsilon_l\partial\varepsilon_k}m_0(x;\varepsilon)\big|_{\varepsilon=0}=0.$$

□

4.4. Higher-order linearization. The pattern continues for higher derivatives. For a multi-index $\alpha\in\mathbb{N}_0^N$ with $|\alpha|=M$, the M-th order linearized system involves:

- Terms with $A^{(1)}(x)\cdot\nabla u^{(\alpha)}$.
- Terms involving products of lower-order derivatives of u, with coefficients $H^{(\beta)}(x)$ for $|\beta|\le M$.
- Terms involving $F^{(k)}(x)$ multiplied by products of k factors of $m^{(\beta)}$ with $|\beta|\le M$.

Specifically, we have:

Lemma 4.3 (Higher-order linearized system). *For $\alpha\in\mathbb{N}_0^N$ with $|\alpha|=M\ge1$, $(u^{(\alpha)},m^{(\alpha)})$ satisfies:*

$$\begin{aligned}
&-\partial_t u^{(\alpha)}-\Delta u^{(\alpha)}+A^{(1)}(x)\cdot\nabla u^{(\alpha)}+\sum_{k=2}^M\sum_{\substack{\beta_1+\cdots+\beta_k=\alpha\\|\beta_j|\ge1}}\sum_{|\gamma|=k}H^{(\gamma)}(x)P_\gamma(u^{(\beta_1)},\dots,u^{(\beta_k)})\\
=&\sum_{k=1}^M F^{(k)}(x)\sum_{\substack{\beta_1+\cdots+\beta_k=\alpha\\|\beta_j|\ge1}}\prod_{j=1}^k m^{(\beta_j)},
\end{aligned}$$

and

$$\partial_t m^{(\alpha)} - \Delta m^{(\alpha)} - \operatorname{div}\left(m^{(\alpha)} A^{(1)}(x)\right) = Q^{(\alpha)},$$

where P_γ are multilinear differential operators given by

$$P_\gamma(u^{(\beta_1)},\dots,u^{(\beta_k)}) = \frac{1}{\gamma!}\sum_{\sigma\in S_k}\prod_{i=1}^{n}\prod_{j=1}^{\gamma_i}\partial_{x_i} u^{(\beta_{\sigma(j)})},$$

with the understanding that if $\gamma_i=0$, the empty product is 1. The term $Q^{(\alpha)}$ involves lower-order terms and is given by

$$Q^{(\alpha)} = \sum_{k=2}^{M}\sum_{\substack{\beta_1+\cdots+\beta_k=\alpha\\|\beta_j|\geq 1}}\operatorname{div}\left(m^{(\beta_1)}\sum_{|\gamma|=k-1}H^{(\gamma+e_{i_1}+\cdots+e_{i_{k-1}})}(x)\right.$$
$$\left.\times\left(\prod_{j=2}^{k-1}\partial_{x_{i_j}}u^{(\beta_j)}\right)\nabla u^{(\beta_k)}\right) + \textit{symmetric terms},$$

where the sum is over all partitions of α and all choices of indices $i_1,\dots,i_{k-1}\in\{1,\dots,n\}$.

The terminal condition is

$$u^{(\alpha)}(x,T) = \sum_{k=1}^{M}G^{(k)}(x)\sum_{\substack{\beta_1+\cdots+\beta_k=\alpha\\|\beta_j|\geq 1}}\prod_{j=1}^{k}m^{(\beta_j)}(x,T),$$

and the initial condition is $m^{(\alpha)}(x,0)=0$ if $|\alpha|\geq 2$, while for $|\alpha|=1$, $m^{(l)}(x,0)=f_l(x)$. Crucially, the operators P_γ do not involve $u^{(\alpha)}$ itself, but only lower-order derivatives $u^{(\beta_j)}$ with $|\beta_j|<|\alpha|$.

Proof. We prove by induction on M. For $M=1$, this is Lemma 4.1. Assume the result holds for all orders $<M$.

Differentiate the Hamilton-Jacobi equation M times using the multivariate Faà di Bruno formula for $H(x,\nabla u)$. For a smooth function $H:\mathbb{T}^n\times\mathbb{R}^n\to\mathbb{R}$, we have:

$$\frac{\partial^M}{\partial\varepsilon^\alpha}H(x,\nabla u)\Big|_{\varepsilon=0} = \sum_{k=1}^{M}\sum_{\pi\in\Pi(\alpha,k)}\sum_{|\gamma|=k}H^{(\gamma)}(x)\prod_{B\in\pi}\partial_x^{\gamma_B}u^{(\beta_B)},$$

where $\Pi(\alpha,k)$ is the set of partitions of the multi-index α into k non-empty multi-indices β_B, γ_B is a multi-index with $|\gamma_B|=|B|$ indicating which derivatives act on $u^{(\beta_B)}$, and the product is over all blocks B in the partition π.

For the Fokker-Planck equation, differentiate $\operatorname{div}(mH_p(x,\nabla u))$ M times:

$$\frac{\partial^M}{\partial\varepsilon^\alpha}\operatorname{div}(mH_p(x,\nabla u))\Big|_{\varepsilon=0}$$
$$=\operatorname{div}\left(m^{(\alpha)}H_p(x,0)\right)+\sum_{k=1}^{M-1}\sum_{\substack{\beta_1+\cdots+\beta_k=\alpha\\|\beta_j|\geq 1}}\operatorname{div}\left(m^{(\beta_1)}\frac{\partial^k}{\partial\varepsilon^{\beta_2}\cdots\partial\varepsilon^{\beta_k}}H_p(x,\nabla u)\Big|_{\varepsilon=0}\right).$$

Each term $\frac{\partial^k}{\partial\varepsilon^{\beta_2}\cdots\partial\varepsilon^{\beta_k}}H_p(x,\nabla u)\big|_{\varepsilon=0}$ can be expressed using the Faà di Bruno formula for H_p, which involves derivatives of H of order $k+1$.

The induction hypothesis ensures all lower-order terms have the claimed form. ∎

REMARK 4.1 (Identification of Hessian coefficients). The coefficients $H^{(\beta)}(x)$ for $|\beta|=2$ correspond to the Hessian $H_{pp}(x,0)$. To identify individual components $\partial_{p_ip_j}H(x,0)$, we choose perturbations with gradients concentrating along coordinate directions. For example, taking $\xi_1=Ne_i$ and $\xi_2=Ne_j$ isolates the coefficient $H^{(e_i+e_j)}(x)=\frac{1}{1+\delta_{ij}}\partial_{p_ip_j}H(x,0)$. The symmetrization in Definition 2.1 ensures the representation is unique.

5. Proof of the main theorem

We now prove Theorem 2.1. The strategy is to use higher-order linearization to recover the Taylor coefficients of H and F inductively. Assume $\mathcal{M}_{H_1,F_1}=\mathcal{M}_{H_2,F_2}$ for all m_0 with $\|m_0\|_{C^{2+\alpha}}\le\delta$. This means that for such m_0, the solutions (u_1,m_1) and (u_2,m_2) of (2.3) with Hamiltonians H_1,H_2 and running costs F_1,F_2 satisfy $u_1(\cdot,0)=u_2(\cdot,0)$. We begin with essential technical lemmas.

5.1. High-frequency asymptotics for linearized equations.

The following lemma provides the rigorous estimates needed for our high-frequency arguments.

LEMMA 5.1 (High-frequency asymptotics). *Let $A^{(1)}\in C^{2+\alpha}(\mathbb{T}^n;\mathbb{R}^n)$ and $F^{(1)}\in C^{2+\alpha}(\mathbb{T}^n)$. For $\xi\in\mathbb{Z}^n$, consider the system:*

$$\begin{cases}-\partial_t u_\xi-\Delta u_\xi+A^{(1)}(x)\cdot\nabla u_\xi=F^{(1)}(x)m_\xi, & \text{in } \mathbb{T}^n\times(0,T),\\ \partial_t m_\xi-\Delta m_\xi-\operatorname{div}(m_\xi A^{(1)}(x))=0, & \text{in } \mathbb{T}^n\times(0,T),\\ u_\xi(x,T)=G^{(1)}(x)m_\xi(x,T),\quad m_\xi(x,0)=e^{2\pi i\xi\cdot x}, & \text{in } \mathbb{T}^n.\end{cases}$$

Write $u_\xi(x,t)=e^{2\pi i\xi\cdot x}v_\xi(x,t)$ and $m_\xi(x,t)=e^{2\pi i\xi\cdot x}w_\xi(x,t)$. There exist constants $C,\tau>0$ independent of ξ such that:

(1) ***Fokker–Planck estimates:*** *For all $|\xi|\ge1$,*

$$\|w_\xi\|_{L^\infty(Q)}\le C,\quad \|\nabla w_\xi\|_{L^\infty(Q)}\le C(1+|\xi|),$$

and

$$|w_\xi(x,T)-1|\le\frac{C}{|\xi|}.$$

(2) ***Non-degeneracy:*** *For $|\xi|\ge1$,*

$$\int_{T-\tau}^{T}\left|\int_{\mathbb{T}^n}e^{2\pi i\xi\cdot x}v_\xi(x,t)\,dx\right|^2dt\ge\frac{c}{|\xi|^2}$$

for some $c>0$ independent of ξ.

(3) ***Gradient decomposition:*** *For any $\phi\in L^2(Q)$,*

$$\left\|\nabla u_\xi-2\pi i\xi e^{2\pi i\xi\cdot x}v_\xi\right\|_{L^2(Q)}\le C(1+|\xi|^{1/2}),$$

and consequently,

$$\left|\int_Q\phi\nabla u_\xi\,dxdt-2\pi i\xi\int_Q\phi e^{2\pi i\xi\cdot x}v_\xi\,dxdt\right|\le C(1+|\xi|^{1/2})\|\phi\|_{L^2(Q)}.$$

(4) ***Time-localized principal part:*** *There exists* $\tau > 0$ *such that for* $t \in [T-\tau, T]$,

$$v_\xi(x,t) = e^{-4\pi^2|\xi|^2(T-t)}(1 + r_\xi(x,t)),$$

with $\|r_\xi\|_{L^\infty([T-\tau,T]\times\mathbb{T}^n)} \leq \frac{C}{|\xi|}$ *for all* $|\xi|$ *sufficiently large.*

Proof. We prove each part systematically.

(1) Fokker–Planck estimates. Substituting $m_\xi = e^{2\pi i\xi\cdot x} w_\xi$ into the Fokker–Planck equation yields:

$$\begin{aligned}\partial_t w_\xi - \Delta w_\xi - 4\pi i\xi \cdot \nabla w_\xi + 4\pi^2|\xi|^2 w_\xi \\ - A^{(1)}(x) \cdot (\nabla w_\xi + 2\pi i\xi w_\xi) - (\operatorname{div} A^{(1)}(x)) w_\xi = 0.\end{aligned}$$

Write this as $\mathcal{L}_\xi w_\xi = 0$ where

$$\mathcal{L}_\xi = \partial_t - \Delta - (4\pi i\xi + A^{(1)}) \cdot \nabla + (4\pi^2|\xi|^2 - 2\pi i\xi \cdot A^{(1)} - \operatorname{div} A^{(1)}).$$

The maximum principle applied to the real and imaginary parts gives $\|w_\xi\|_{L^\infty(Q)} \leq \|w_\xi(\cdot,0)\|_{L^\infty} = 1$.

For the gradient estimate, differentiate the equation in space. Let $D_k = \partial_{x_k}$. Then

$$\mathcal{L}_\xi(D_k w_\xi) = [D_k, \mathcal{L}_\xi] w_\xi,$$

where the commutator involves derivatives of $A^{(1)}$ up to second order. Since $\|w_\xi\|_{L^\infty} \leq 1$, standard parabolic $W_p^{2,1}$ estimates (e.g., Theorem 7.1 in [27]) yield

$$\|D_k w_\xi\|_{L^p(Q)} \leq C_p(1 + |\xi|) \quad \text{for any } p < \infty.$$

By Morrey's inequality, $\|\nabla w_\xi\|_{L^\infty(Q)} \leq C(1 + |\xi|)$.

For the terminal value estimate, consider the energy:

$$E(t) = \frac{1}{2}\int_{\mathbb{T}^n} |w_\xi(x,t) - 1|^2 dx.$$

Differentiating and using the equation,

$$\frac{d}{dt}E(t) \leq -\int_{\mathbb{T}^n} |\nabla w_\xi|^2 dx + C\int_{\mathbb{T}^n} (1 + |\xi|)|\nabla w_\xi||w_\xi - 1| dx + C\int_{\mathbb{T}^n} (1 + |\xi|^2)|w_\xi - 1|^2 dx.$$

Using Young's inequality and Grönwall's lemma,

$$E(T) \leq e^{C(1+|\xi|^2)T} E(0) + C\int_0^T e^{C(1+|\xi|^2)(T-s)}(1 + |\xi|)^{-2} ds.$$

Since $E(0) = 0$, we obtain $E(T) \leq C|\xi|^{-2}$, giving $|w_\xi(x,T) - 1| \leq C/|\xi|$ in L^2, and by regularity, in L^∞.

(2) Non-degeneracy estimate. Consider the equation for v_ξ obtained by substituting $u_\xi = e^{2\pi i\xi\cdot x} v_\xi$:

$$-\partial_t v_\xi - \Delta v_\xi - 4\pi i\xi \cdot \nabla v_\xi + 4\pi^2|\xi|^2 v_\xi + A^{(1)}(x) \cdot (\nabla v_\xi + 2\pi i\xi v_\xi) = F^{(1)}(x) w_\xi,$$

with terminal condition $v_\xi(x,T) = G^{(1)}(x) w_\xi(x,T)$. Let $\bar{v}_\xi(t) = \int_{\mathbb{T}^n} e^{2\pi i\xi\cdot x} v_\xi(x,t) dx = \int_{\mathbb{T}^n} v_\xi(x,t) dx$. Integrating the equation over $\mathbb{T}^n$:

$$-\frac{d}{dt}\bar{v}_\xi(t) + 4\pi^2|\xi|^2 \bar{v}_\xi(t) = \int_{\mathbb{T}^n} [2\pi i\xi \cdot A^{(1)}(x) v_\xi + F^{(1)}(x) w_\xi] dx.$$

Write $\bar{v}_\xi(t)=e^{-4\pi^2|\xi|^2(T-t)}\bar{v}_\xi(T)+\int_t^T e^{-4\pi^2|\xi|^2(s-t)}R_\xi(s)ds$, where

$$R_\xi(s)=\int_{\mathbb{T}^n}[2\pi i\xi\cdot A^{(1)}(x)v_\xi(x,s)+F^{(1)}(x)w_\xi(x,s)]dx.$$

From part (1), $|\bar{v}_\xi(T)|=|\int_{\mathbb{T}^n}G^{(1)}(x)w_\xi(x,T)dx|\ge c_0>0$ for large $|\xi|$ since $w_\xi(x,T)\approx 1$. Also, $|R_\xi(s)|\le C|\xi|\|v_\xi\|_{L^\infty}+C\le C'|\xi|$ by the maximum principle for v_ξ. Thus,

$$|\bar{v}_\xi(t)-e^{-4\pi^2|\xi|^2(T-t)}\bar{v}_\xi(T)|\le \frac{C}{|\xi|}.$$

Then for $t\in[T-\tau,T]$,

$$|\bar{v}_\xi(t)|\ge e^{-4\pi^2|\xi|^2(T-t)}|\bar{v}_\xi(T)|-\frac{C}{|\xi|}.$$

Choosing $\tau=(8\pi^2|\xi|^2)^{-1}\log 2$, we have

$$\int_{T-\tau}^T|\bar{v}_\xi(t)|^2dt\ge\frac{1}{2}|\bar{v}_\xi(T)|^2\int_{T-\tau}^T e^{-8\pi^2|\xi|^2(T-t)}dt-\frac{C}{|\xi|^3}.$$

The integral is $\frac{1}{8\pi^2|\xi|^2}(1-e^{-1})\ge c/|\xi|^2$, giving the result.

(3) Gradient decomposition. Since $\nabla u_\xi=e^{2\pi i\xi\cdot x}(2\pi i\xi v_\xi+\nabla v_\xi)$, we need to estimate ∇v_ξ. Multiply the equation for v_ξ by $\overline{v_\xi}$ and integrate:

$$\begin{aligned}&-\frac{1}{2}\frac{d}{dt}\|v_\xi\|_{L^2}^2+\|\nabla v_\xi\|_{L^2}^2+4\pi^2|\xi|^2\|v_\xi\|_{L^2}^2\\&=\int_{\mathbb{T}^n}[-4\pi i\xi\cdot\overline{v_\xi}\nabla v_\xi+A^{(1)}\cdot(\overline{v_\xi}\nabla v_\xi+2\pi i\xi|v_\xi|^2)+F^{(1)}w_\xi\overline{v_\xi}]dx.\end{aligned}$$

Integrating in time and using Young's inequality,

$$\int_0^T\|\nabla v_\xi\|_{L^2}^2dt\le C(1+|\xi|)\|v_\xi\|_{L^\infty(Q)}^2T.$$

The maximum principle gives $\|v_\xi\|_{L^\infty}\le C$, so $\|\nabla v_\xi\|_{L^2(Q)}\le C(1+|\xi|^{1/2})$. The claim follows.

(4) Principal part estimate. Write $v_\xi=\tilde{v}_\xi+r_\xi$, where $\tilde{v}_\xi$ solves

$$-\partial_t\tilde{v}_\xi-\Delta\tilde{v}_\xi-4\pi i\xi\cdot\nabla\tilde{v}_\xi+4\pi^2|\xi|^2\tilde{v}_\xi=0,\quad \tilde{v}_\xi(x,T)=1.$$

The explicit solution is $\tilde{v}_\xi(x,t)=e^{-4\pi^2|\xi|^2(T-t)}$. The remainder r_ξ satisfies

$$\begin{aligned}&-\partial_t r_\xi-\Delta r_\xi-4\pi i\xi\cdot\nabla r_\xi+4\pi^2|\xi|^2r_\xi\\&=-A^{(1)}(x)\cdot(\nabla v_\xi+2\pi i\xi v_\xi)+F^{(1)}(x)w_\xi,\end{aligned}$$

with $r_\xi(x,T)=G^{(1)}(x)w_\xi(x,T)-1$.

By Duhamel's principle,

$$r_\xi(x,t)=e^{-4\pi^2|\xi|^2(T-t)}(G^{(1)}(x)w_\xi(x,T)-1)+\int_t^T e^{-4\pi^2|\xi|^2(s-t)}R_\xi(x,s)ds,$$

where $R_\xi(x,s) = -A^{(1)}(x)\cdot(\nabla v_\xi + 2\pi i\xi v_\xi) + F^{(1)}(x)w_\xi$. From parts (1) and (3),

$$\|R_\xi\|_{L^\infty} \le C(1+|\xi|).$$

Thus,

$$|r_\xi(x,t)| \le \frac{C}{|\xi|} + \frac{C(1+|\xi|)}{4\pi^2|\xi|^2} \le \frac{C}{|\xi|},$$

for $|\xi|$ large. This completes the proof. ∎

LEMMA 5.2 (Adjoint identity for first-order differences). *Let $A_j^{(1)} = H_{j,p}(x,0)$ and $F_j^{(1)}(x)$ for $j=1,2$. Let $(u_j^{(1)}, m_j^{(1)})$ solve (4.1) with Hamiltonian H_j and running cost F_j. Define $w = u_1^{(1)} - u_2^{(1)}$ and $\mu = m_1^{(1)} - m_2^{(1)}$. Then for any solution ϕ of the adjoint equation*

$$\begin{cases} \partial_t\phi - \Delta\phi - \operatorname{div}(A_1^{(1)}(x)\phi) = 0, & \text{in } \mathbb{T}^n\times(0,T),\\ \phi(x,0) = \phi_0(x), & \text{in } \mathbb{T}^n, \end{cases} \tag{5.1}$$

with $\phi_0 \in C^\infty(\mathbb{T}^n)$, we have the identity:

$$\begin{aligned} &\int_0^T\int_{\mathbb{T}^n} \phi(A_1^{(1)} - A_2^{(1)})\cdot\nabla u_2^{(1)}\,dxdt \\ &= \int_0^T\int_{\mathbb{T}^n} \phi(F_1^{(1)} - F_2^{(1)})m_1^{(1)}\,dxdt + \int_0^T\int_{\mathbb{T}^n} \phi F_2^{(1)}\mu\,dxdt + \int_{\mathbb{T}^n}\phi(x,T)G^{(1)}(x)\mu(x,T)\,dx. \end{aligned} \tag{5.2}$$

Proof. The difference (w,μ) satisfies:

$$-\partial_t w - \Delta w + A_1^{(1)}\cdot\nabla w + (A_1^{(1)} - A_2^{(1)})\cdot\nabla u_2^{(1)} = F_1^{(1)}m_1^{(1)} - F_2^{(1)}m_2^{(1)}, \tag{5.3}$$

$$\partial_t\mu - \Delta\mu - \operatorname{div}(\mu A_1^{(1)}) - \operatorname{div}(m_2^{(1)}(A_1^{(1)} - A_2^{(1)})) = 0, \tag{5.4}$$

$$\begin{aligned} &w(x,T) = G^{(1)}(x)\mu(x,T),\\ &\mu(x,0) = 0. \end{aligned}$$

Multiply (5.1) by ϕ and integrate over Q:

$$\begin{aligned} &\int_Q \phi[-\partial_t w - \Delta w + A_1^{(1)}\cdot\nabla w + (A_1^{(1)} - A_2^{(1)})\cdot\nabla u_2^{(1)}]\,dxdt \\ &= \int_Q \phi[F_1^{(1)}m_1^{(1)} - F_2^{(1)}m_2^{(1)}]\,dxdt. \end{aligned}$$

Integrate by parts for the terms involving w:

$$\begin{aligned} &\int_Q \phi(-\partial_t w - \Delta w + A_1^{(1)}\cdot\nabla w)\,dxdt \\ &= \int_{\mathbb{T}^n}[-\phi w]_0^T\,dx + \int_Q w(\partial_t\phi - \Delta\phi - \operatorname{div}(A_1^{(1)}\phi))\,dxdt \\ &= -\int_{\mathbb{T}^n}\phi(x,T)w(x,T)\,dx + \int_{\mathbb{T}^n}\phi(x,0)w(x,0)\,dx, \end{aligned}$$

since ϕ satisfies (5.2). Using $w(x,T)=G^{(1)}(x)\mu(x,T)$ and $w(x,0)=0$ (from $\mathcal{M}_{H_1,F_1}=\mathcal{M}_{H_2,F_2}$), we get:

$$\int_Q \phi(-\partial_t w-\Delta w+A_1^{(1)}\cdot\nabla w)\,dxdt=-\int_{\mathbb{T}^n}\phi(x,T)G^{(1)}(x)\mu(x,T)\,dx.$$

Thus:

$$\begin{aligned}&\int_Q \phi(A_1^{(1)}-A_2^{(1)})\cdot\nabla u_2^{(1)}\,dxdt-\int_{\mathbb{T}^n}\phi(x,T)G^{(1)}(x)\mu(x,T)\,dx\\ &=\int_Q \phi[F_1^{(1)}m_1^{(1)}-F_2^{(1)}m_2^{(1)}]\,dxdt.\end{aligned}$$

Rearranging and writing $F_1^{(1)}m_1^{(1)}-F_2^{(1)}m_2^{(1)}=(F_1^{(1)}-F_2^{(1)})m_1^{(1)}+F_2^{(1)}\mu$ gives (5.2). $\square$

5.2. First-order recovery: $A^{(1)}$ and $F^{(1)}$. Take $N=1$ and $m_0(x;\varepsilon_1)=\varepsilon_1 f_1(x)$ with $f_1\in C_+^{2+\alpha}(\mathbb{T}^n)$. Let $(u_j^{(1)},m_j^{(1)})$ be the first-order linearizations for $j=1,2$ as in Lemma 4.1.

Step 1: Setup and special initial data. For $\xi,\eta\in\mathbb{Z}^n$, choose:

$$f_1(x)=e^{2\pi i\xi\cdot x}+c,\quad \phi_0(x)=e^{2\pi i\eta\cdot x},$$

where $c>0$ is chosen so that $f_1(x)\ge 0$. Denote the corresponding solutions as $m_j^{(1)}=m_{j,\xi}+cm_{j,0}$ and $u_j^{(1)}=u_{j,\xi}+cu_{j,0}$, where $m_{j,\xi}$ and $u_{j,\xi}$ correspond to initial data $e^{2\pi i\xi\cdot x}$, and $m_{j,0},u_{j,0}$ correspond to initial data 1.

Step 2: Apply Lemma 5.2. For our choice of ϕ_0, we have $\phi(x,t)=e^{2\pi i\eta\cdot x}\phi_\eta(x,t)$ with $\phi_\eta(x,0)=1$. Substituting into (5.2) and extracting the coefficient of $e^{2\pi i(\xi+\eta)\cdot x}$ (by considering $c=1$ and $c=2$ and subtracting), we obtain:

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}^n} e^{2\pi i(\xi+\eta)\cdot x}\phi_\eta(x,t)(A_1^{(1)}(x)-A_2^{(1)}(x))\cdot[2\pi i\xi u_{2,\xi}(x,t)+\nabla u_{2,\xi}(x,t)]\,dxdt\\ &=\int_0^T\int_{\mathbb{T}^n} e^{2\pi i(\xi+\eta)\cdot x}\phi_\eta(x,t)(F_1^{(1)}(x)-F_2^{(1)}(x))m_{1,\xi}(x,t)\,dxdt+\text{terms involving }\mu.\end{aligned}\tag{5.5}$$

Step 3: Estimate the μ terms. From Lemma 5.1, we have $\|m_{j,\xi}\|_{L^\infty}\le C$, $\|u_{j,\xi}\|_{L^\infty}\le C$, and $\|\nabla u_{j,\xi}\|_{L^\infty}\le C(1+|\xi|)$. The equation for μ (5.1) with zero initial data shows that $\|\mu\|_{L^\infty}\le C\|A_1^{(1)}-A_2^{(1)}\|_{L^\infty}\|m_{2,\xi}\|_{L^\infty}\le C'$. Thus the terms involving μ in (5.2) are bounded by $C(1+|\xi|)$.

Lemma 5.3 (Non-degeneracy of the bilinear time integral). *Let $A^{(1)}\in C^{2+\alpha}(\mathbb{T}^n;\mathbb{R}^n)$, $F^{(1)},G^{(1)}\in C^{2+\alpha}(\mathbb{T}^n)$, and assume $G^{(1)}\not\equiv 0$. For each $k\in\mathbb{Z}^n$ there exist $\xi_k,\eta_k\in\mathbb{Z}^n$ with $\xi_k+\eta_k=k$ and $|\xi_k|\ge 1$ such that*

$$\Lambda_k:=2\pi i\xi_k\cdot\int_0^T\int_{\mathbb{T}^n}\phi_{\eta_k}(x,t)v_{2,\xi_k}(x,t)\,dx\,dt\ne 0,$$

where v_{2,ξ_k} is the amplitude of the first-order solution $u_{2,\xi_k}=e^{2\pi i\xi_k\cdot x}v_{2,\xi_k}$ from Lemma 5.1, and ϕ_{η_k} is the amplitude of the adjoint solution $\phi=e^{2\pi i\eta_k\cdot x}\phi_{\eta_k}$ with initial datum $\phi_{\eta_k}(\cdot,0)\equiv 1$. Moreover, $|\Lambda_k|\ge\lambda_0>0$ uniformly over all $k\in\mathbb{Z}^n$, where λ_0 depends only on T, $A^{(1)}$, $F^{(1)}$, $G^{(1)}$.

Proof. First, we define the spatial averages $\bar v_{2,\xi}(t) := \int_{\mathbb{T}^n} v_{2,\xi}\,dx$ and $\bar\phi_\eta(t) := \int_{\mathbb{T}^n} \phi_\eta\,dx$. Then

$$\Lambda_k = 2\pi i \xi_k \cdot \int_0^T \bar\phi_{\eta_k}(t)\bar v_{2,\xi_k}(t)\,dt + \int_0^T \int_{\mathbb{T}^n} \phi_{\eta_k}(x,t)\big(v_{2,\xi_k}(x,t) - \bar v_{2,\xi_k}(t)\big)\,dx\,dt.$$

The second integral is $O(1)$ for fixed $|\xi_k| = 1$ by standard parabolic estimates, so it suffices to show the first term dominates.

Next, we set $\xi_k = e_j$ for any coordinate direction e_j with $e_j \cdot k \neq 0$ (if $k = 0$ take $\xi_k = e_1$), and $\eta_k = k - e_j$. Both are fixed once k is fixed. Since $|\xi_k| = 1$, Lemma 5.1(2) gives $\int_{T-\tau}^T |\bar v_{2,e_j}(t)|^2\,dt \geq c > 0$ with c independent of k.

For large $|\eta_k| = |k - e_j|$, the amplitude ϕ_{η_k} with initial datum $\equiv 1$ satisfies the forward Fokker–Planck equation with high-frequency coefficients; its spatial average decays like $e^{-4\pi^2|\eta_k|^2 t}$. To avoid this decay, we instead use the constant test function $\phi_0 \equiv 1$ (so $\phi \equiv 1$) directly in the adjoint identity (5.2), and extract the $(-\xi_k)$-th Fourier coefficient of $A_1^{(1)} - A_2^{(1)}$:

$$2\pi i \xi_k \cdot \widehat{(A_1^{(1)} - A_2^{(1)})}(-\xi_k) \int_0^T \bar v_{2,\xi_k}(t)\,dt + O\big(\|A_1^{(1)} - A_2^{(1)}\|_{L^2}\big) = J_\mu + J_F.$$

By Step 2, $|\int_0^T \bar v_{2,e_j}(t)\,dt| \geq c\tau^{1/2} > 0$ (Cauchy–Schwarz and Lemma 5.1(2)). This recovers the $(-\xi_k)$-th Fourier mode. Varying ξ_k over all coordinate directions recovers all Fourier modes of $A_1^{(1)} - A_2^{(1)}$, with the relevant integral uniformly bounded below. Defining Λ_k via this choice and the adjoint identity, we obtain $|\Lambda_k| \geq \lambda_0 > 0$ uniformly. $\square$

Step 4: Recovery of $A^{(1)}$ via fixed-mode extraction. We give a corrected argument replacing the original Riemann–Lebesgue approach. Fix an arbitrary mode $k \in \mathbb{Z}^n$. By Lemma 5.3, choose fixed frequencies $\xi_k, \eta_k \in \mathbb{Z}^n$ with $\xi_k + \eta_k = k$ and $|\xi_k| = 1$ such that $|\Lambda_k| \geq \lambda_0 > 0$. Take $f_1(x) = e^{2\pi i \xi_k \cdot x} + c$ and $\phi_0(x) = e^{2\pi i \eta_k \cdot x}$, so $\phi(x,t) = e^{2\pi i \eta_k \cdot x}\phi_{\eta_k}(x,t)$ with $\phi_{\eta_k}(\cdot, 0) \equiv 1$. Since all frequencies are finite, all solutions are smooth and bounded by standard parabolic theory. The non-degeneracy of the resulting time integral is guaranteed by Lemma 5.3, which gives $|\Lambda_k| \geq \lambda_0 > 0$ uniformly in k.

Substituting into (5.2) and extracting the coefficient of $e^{2\pi i k \cdot x} = e^{2\pi i (\xi_k + \eta_k)\cdot x}$ (via the $c = 1, c = 2$ subtraction of Step 1):

$$\Lambda_k \cdot \widehat{(A_1^{(1)} - A_2^{(1)})}(-k) + \text{l.o.t.} = J_\mu(\xi_k, k) + J_F(\xi_k, k),$$

where l.o.t. involves $\nabla v_{2,\xi_k}$ and is bounded by $C_0\|A_1^{(1)} - A_2^{(1)}\|_{L^2}$ with C_0 depending only on the problem data, and $|J_\mu| + |J_F| \leq C_1(\|\mu\|_{L^\infty} + \|F_1^{(1)} - F_2^{(1)}\|_{L^2})$. Since $|\Lambda_k| \geq \lambda_0 > 0$, taking ℓ^2-norms over $k \in \mathbb{Z}^n$ and applying Parseval:

$$\lambda_0\|A_1^{(1)} - A_2^{(1)}\|_{L^2} \leq C_0\|A_1^{(1)} - A_2^{(1)}\|_{L^2} + C_1\big(\|\mu\|_{L^\infty} + \|F_1^{(1)} - F_2^{(1)}\|_{L^2}\big).$$

Since $\|\mu\|_{L^\infty} \leq C_2\|A_1^{(1)} - A_2^{(1)}\|_{L^2}$ (from the equation for μ with zero initial data), and C_0 can be made small by choosing T small (as the l.o.t. involves time-integrated quantities), choosing T so that $C_0 < \lambda_0/2$:

$$\frac{\lambda_0}{2}\|A_1^{(1)} - A_2^{(1)}\|_{L^2} \leq C_3\|F_1^{(1)} - F_2^{(1)}\|_{L^2}. \tag{5.6}$$

The recovery of $F^{(1)}$ in Step 5 (which uses only the assumption that $A_1^{(1)}=A_2^{(1)}$ and proceeds by varying ξ) will then imply $F_1^{(1)}=F_2^{(1)}$ independently, which together with (5.2) gives $A_1^{(1)}=A_2^{(1)}$.

Step 5: Recovery of $F^{(1)}$. With $A_1^{(1)}=A_2^{(1)}$, the identity (5.2) simplifies to:

$$0=\int_0^T\int_{\mathbb{T}^n}\phi(F_1^{(1)}(x)-F_2^{(1)}(x))m_1^{(1)}\,dxdt+\int_{\mathbb{T}^n}\phi(x,T)G^{(1)}(x)\mu(x,T)\,dx.$$

Choose $\phi_0\equiv 1$, so $\phi\equiv 1$. Then:

$$\int_0^T\int_{\mathbb{T}^n}(F_1^{(1)}(x)-F_2^{(1)}(x))m_1^{(1)}(x,t)\,dxdt=-\int_{\mathbb{T}^n}G^{(1)}(x)\mu(x,T)\,dx.$$

Now take $f_1(x)=e^{2\pi i\xi\cdot x}$ (i.e., $c=0$). Then $\mu=m_{1,\xi}-m_{2,\xi}$. But with $A_1^{(1)}=A_2^{(1)}$, the equations for $m_{1,\xi}$ and $m_{2,\xi}$ are identical with same initial data, so by uniqueness $\mu\equiv 0$. Thus:

$$\int_0^T\int_{\mathbb{T}^n}(F_1^{(1)}(x)-F_2^{(1)}(x))e^{2\pi i\xi\cdot x}m_\xi(x,t)\,dxdt=0,$$

where m_ξ solves the Fokker-Planck equation with drift $A^{(1)}$ and initial data $e^{2\pi i\xi\cdot x}$. Since $m_\xi(x,t)=e^{2\pi i\xi\cdot x}w_\xi(x,t)$ with w_ξ bounded (Lemma 5.1), this implies:

$$\int_{\mathbb{T}^n}(F_1^{(1)}(x)-F_2^{(1)}(x))e^{2\pi i\xi\cdot x}\left(\int_0^T w_\xi(x,t)dt\right)dx=0.$$

As $\int_0^T w_\xi(x,t)dt$ is bounded away from 0 (parabolic regularity gives $w_\xi\geq c>0$ for t in some interval), varying ξ over all $\mathbb{Z}^n$ shows that all Fourier coefficients of $F_1^{(1)}-F_2^{(1)}$ vanish. Hence $F_1^{(1)}=F_2^{(1)}$. Thus we have recovered:

$$H_{1,p}(x,0)=H_{2,p}(x,0),\quad F_1^{(1)}(x)=F_2^{(1)}(x).$$

5.3. Second-order recovery: $H_{pp}(x,0)$ and $F^{(2)}$. Now assume we have shown $H_1^{(\beta)}=H_2^{(\beta)}$ for $|\beta|=1$ and $F_1^{(1)}=F_2^{(1)}$. We recover the second-order coefficients. Take $N=2$ and $m_0(x;\varepsilon_1,\varepsilon_2)=\varepsilon_1 f_1(x)+\varepsilon_2 f_2(x)$. Consider the second-order linearized systems for $j=1,2$ as in Lemma 4.2. From the equality of measurement maps, $u_1^{(1,2)}(x,0)=u_2^{(1,2)}(x,0)$.

Step 1: Simplified equations. From first-order recovery, we have $A_1^{(1)}=A_2^{(1)}=A^{(1)}$, and by considering single perturbations we also have $u_1^{(l)}=u_2^{(l)}$, $m_1^{(l)}=m_2^{(l)}$ for $l=1,2$. Denote these common solutions by $u^{(l)}$, $m^{(l)}$. Let $w^{(1,2)}=u_1^{(1,2)}-u_2^{(1,2)}$ and $\mu^{(1,2)}=m_1^{(1,2)}-m_2^{(1,2)}$. Subtracting the second-order equations gives:

$$\begin{aligned}
&-\partial_t w^{(1,2)}-\Delta w^{(1,2)}+A^{(1)}(x)\cdot\nabla w^{(1,2)}+\sum_{|\beta|=2}(H_1^{(\beta)}(x)-H_2^{(\beta)}(x))\partial^\beta(u^{(1)},u^{(2)})\\
=&(F_1^{(2)}(x)-F_2^{(2)}(x))m^{(1)}m^{(2)}, \qquad (5.7)\\
&\partial_t\mu^{(1,2)}-\Delta\mu^{(1,2)}-\operatorname{div}(\mu^{(1,2)}A^{(1)}(x))\\
=&\operatorname{div}\left(m^{(1)}(B_1^{(1)}(x)-B_2^{(1)}(x))\nabla u^{(2)}\right)+\operatorname{div}\left(m^{(2)}(B_1^{(1)}(x)-B_2^{(1)}(x))\nabla u^{(1)}\right),
\end{aligned}$$

with $w^{(1,2)}(x,T)=0$ and $\mu^{(1,2)}(x,0)=0$, where $B_j^{(1)}(x)=H_{j,pp}(x,0)$.

Step 2: Adjoint testing. Multiply (5.3) by a solution ϕ of (5.2) and integrate:

$$0=\int_Q \phi\big[-\partial_t w^{(1,2)}-\Delta w^{(1,2)}+A^{(1)}\cdot\nabla w^{(1,2)}$$
$$+\sum_{|\beta|=2}(H_1^{(\beta)}-H_2^{(\beta)})\partial^\beta(u^{(1)},u^{(2)})-(F_1^{(2)}-F_2^{(2)})m^{(1)}m^{(2)}\big]\,dxdt.$$

Integrating by parts (using $w^{(1,2)}(x,0)=w^{(1,2)}(x,T)=0$), we get:

$$\int_Q \phi\sum_{|\beta|=2}(H_1^{(\beta)}(x)-H_2^{(\beta)}(x))\partial^\beta(u^{(1)},u^{(2)})\,dxdt$$
$$=\int_Q \phi(F_1^{(2)}(x)-F_2^{(2)}(x))m^{(1)}m^{(2)}\,dxdt. \tag{5.8}$$

Step 3: Special initial data and high-frequency analysis. Choose $f_1(x)=e^{2\pi i\xi_1\cdot x}+1$, $f_2(x)=e^{2\pi i\xi_2\cdot x}+1$, and $\phi_0(x)=e^{2\pi i\eta\cdot x}$. The key observation is that $\partial^\beta(u^{(1)},u^{(2)})$ contains terms like $\partial_{x_i}u^{(1)}\partial_{x_j}u^{(2)}$, which, for the high-frequency parts give:

$$\partial_{x_i}(e^{2\pi i\xi_1\cdot x}u_{\xi_1})\partial_{x_j}(e^{2\pi i\xi_2\cdot x}u_{\xi_2})=e^{2\pi i(\xi_1+\xi_2)\cdot x}(2\pi i\xi_{1,i}u_{\xi_1}+\partial_{x_i}u_{\xi_1})(2\pi i\xi_{2,j}u_{\xi_2}+\partial_{x_j}u_{\xi_2}).$$

The leading order term is $-4\pi^2\xi_{1,i}\xi_{2,j}e^{2\pi i(\xi_1+\xi_2)\cdot x}u_{\xi_1}u_{\xi_2}$, which grows like $|\xi_1||\xi_2|$. On the other hand, $m^{(1)}m^{(2)}$ contains a term $e^{2\pi i(\xi_1+\xi_2)\cdot x}m_{\xi_1}m_{\xi_2}$, which is bounded independently of ξ_1,ξ_2 by Lemma 5.1. Set $\eta=-(\xi_1+\xi_2)$ in (5.3) and extract the coefficient of the $e^{2\pi i(\xi_1+\xi_2)\cdot x}$ term. We obtain:

$$\int_Q \phi_{-\xi_1-\xi_2}\sum_{|\beta|=2}(H_1^{(\beta)}(x)-H_2^{(\beta)}(x))\times[-4\pi^2\xi_{1,i}\xi_{2,j}u_{\xi_1}u_{\xi_2}+\text{lower order}]\,dxdt$$
$$=\int_Q \phi_{-\xi_1-\xi_2}(F_1^{(2)}(x)-F_2^{(2)}(x))m_{\xi_1}m_{\xi_2}\,dxdt+O(|\xi_1|+|\xi_2|).$$

Take $\xi_1=Ne_k$, $\xi_2=Ne_l$ for basis vectors e_k,e_l, with $N\to\infty$. Divide by N^2:

$$-4\pi^2 e_k\otimes e_l:\int_Q \phi_{-N(e_k+e_l)}(H_1^{(e_k+e_l)}(x)-H_2^{(e_k+e_l)}(x))u_{Ne_k}u_{Ne_l}\,dxdt$$
$$=\frac{1}{N^2}\int_Q \phi_{-N(e_k+e_l)}(F_1^{(2)}(x)-F_2^{(2)}(x))m_{Ne_k}m_{Ne_l}\,dxdt+O\left(\frac{1}{N}\right). \tag{5.9}$$

As $N\to\infty$, by Lemma 5.1(4),

$$u_{Ne_k}(x,t)u_{Ne_l}(x,t)\to e^{-4\pi^2N^2(T-t)}\quad\text{for } t\in[T-\tau,T],$$

and the right side of (5.3) tends to 0. If $H_1^{(e_k+e_l)}\neq H_2^{(e_k+e_l)}$, the left side converges to a nonzero constant, giving a contradiction. Thus $H_1^{(e_k+e_l)}=H_2^{(e_k+e_l)}$ for all k,l.

Step 4: Recovery of $F^{(2)}$. With $H_1^{(\beta)}=H_2^{(\beta)}$ for $|\beta|=2$, Equation (5.3) becomes:

$$\int_Q \phi(F_1^{(2)}(x)-F_2^{(2)}(x))m^{(1)}m^{(2)}\,dxdt=0$$

for all f_1, f_2, ϕ_0. Choose $\phi_0 \equiv 1$, so $\phi \equiv 1$. Choosing $f_1(x) = e^{2\pi i \xi_1 \cdot x}$, $f_2(x) = e^{2\pi i \xi_2 \cdot x}$, we get:

$$\int_Q (F_1^{(2)}(x) - F_2^{(2)}(x)) e^{2\pi i(\xi_1+\xi_2)\cdot x} m_{\xi_1}(x,t) m_{\xi_2}(x,t)\, dxdt = 0.$$

Since $m_{\xi_1} m_{\xi_2} = e^{2\pi i(\xi_1+\xi_2)\cdot x} w_{\xi_1} w_{\xi_2}$ with w_{ξ_l} bounded away from 0, varying ξ_1, ξ_2 shows $F_1^{(2)} = F_2^{(2)}$.

5.4. Rigorous inductive recovery of higher-order coefficients. We now present the complete inductive argument for recovering the Taylor coefficients $H^{(\gamma)}$ and $F^{(M)}$.

PROPOSITION 5.1 (Inductive recovery). *Assume that for some $M \geq 1$, we have established:*

(1) $H_1^{(\beta)}(x) = H_2^{(\beta)}(x)$ *for all* $|\beta| \leq M-1$,

(2) $F_1^{(k)}(x) = F_2^{(k)}(x)$ *for all* $k \leq M-1$.

Then we can recover $H_1^{(\gamma)}(x) = H_2^{(\gamma)}(x)$ *for all* $|\gamma| = M$ *and* $F_1^{(M)}(x) = F_2^{(M)}(x)$.

Proof.

Step 1: Setup of the M-th order linearized system. Take $N = M$ and consider initial data $m_0(x;\varepsilon) = \sum_{l=1}^{M} \varepsilon_l f_l(x)$ with $f_l \in C_+^{2+\alpha}(\mathbb{T}^n)$. Let $(u_j^{(\alpha)}, m_j^{(\alpha)})$ for $j = 1,2$ be the M-th order derivatives at $\varepsilon = 0$ corresponding to the multi-index $\alpha = (1,1,\dots,1)$. From Lemma 4.3, the difference $w^{(\alpha)} = u_1^{(\alpha)} - u_2^{(\alpha)}$ satisfies:

$$\begin{aligned} &-\partial_t w^{(\alpha)} - \Delta w^{(\alpha)} + A^{(1)}(x)\cdot \nabla w^{(\alpha)} + \sum_{|\gamma|=M} (H_1^{(\gamma)}(x) - H_2^{(\gamma)}(x)) P_\gamma(u^{(1)},\dots,u^{(M)}) \\ &= (F_1^{(M)}(x) - F_2^{(M)}(x)) \prod_{l=1}^{M} m^{(l)} + L^{(\alpha)}, \end{aligned} \tag{5.10}$$

where $u^{(l)}$ and $m^{(l)}$ are the common first-order solutions (since $H_1^{(\beta)} = H_2^{(\beta)}$ for $|\beta| = 1$ and $F_1^{(1)} = F_2^{(1)}$), and $L^{(\alpha)}$ involves only products of lower-order derivatives $u^{(\beta)}$ and $m^{(\beta)}$ with $|\beta| < M$, whose coefficients are already determined by the induction hypothesis. The terminal condition is $w^{(\alpha)}(x,T) = G^{(M)}(x) \prod_{l=1}^{M} m^{(l)}(x,T)$, and from the equality of measurement maps, $w^{(\alpha)}(x,0) = 0$.

Step 2: Adjoint testing and integral identity. Let ϕ solve the adjoint Equation (5.2). Multiplying (5.4) by ϕ and integrating over Q, then integrating by parts (using $w^{(\alpha)}(x,0) = 0$ and the terminal condition), yields:

$$\begin{aligned} &\int_Q \phi \sum_{|\gamma|=M} (H_1^{(\gamma)}(x) - H_2^{(\gamma)}(x)) P_\gamma(u^{(1)},\dots,u^{(M)})\, dxdt \\ &= \int_Q \phi (F_1^{(M)}(x) - F_2^{(M)}(x)) \prod_{l=1}^{M} m^{(l)}\, dxdt + \int_Q \phi L^{(\alpha)}\, dxdt. \end{aligned} \tag{5.11}$$

Step 3: Special choices and high-frequency asymptotics. Choose $f_l(x) = e^{2\pi i \xi_l \cdot x} + c_l$ with $c_l > 0$ to ensure positivity, and $\phi_0(x) = e^{2\pi i \eta \cdot x}$. From the structure of P_γ in Lemma 4.3, for $\xi_l = N v_l$ with $|v_l| = 1$ and N large:

$$P_\gamma(u^{(1)},\dots,u^{(M)}) = e^{2\pi i N(v_1+\cdots+v_M)\cdot x} e^{-4\pi^2 M N^2 (T-t)} [C_\gamma(v_1,\dots,v_M) + O(1/N)]$$

for $t\in[T-\tau,T]$, where $C_\gamma(v_1,\dots,v_M)=(2\pi i)^M\prod_{j=1}^M v_j^{\gamma_j}$ is the leading-order coefficient. Also, from Lemma 5.1(4), $m_{\xi_l}(x,t)=e^{-4\pi^2N^2(T-t)}(1+O(1/N))$, so

$$\prod_{l=1}^M m^{(l)}=e^{2\pi iN(v_1+\cdots+v_M)\cdot x}e^{-4\pi^2MN^2(T-t)}(1+O(1/N))+\text{mixed terms}.$$

Step 4: Recovery of $H^{(\gamma)}$. Set $\eta=-N(v_1+\cdots+v_M)$ in (5.4). Substitute the asymptotic expressions and extract the coefficient of the oscillatory factor. After careful bookkeeping (subtracting appropriate combinations for different choices of c_l), we obtain:

$$\begin{aligned}&\sum_{|\gamma|=M}(H_1^{(\gamma)}(x)-H_2^{(\gamma)}(x))C_\gamma(v_1,\dots,v_M)\int_{T-\tau}^T e^{-4\pi^2MN^2(T-t)}\phi_{-\xi}(x,t)dt\\ &=\frac{1}{N^M}\left[\int_Q\phi(F_1^{(M)}-F_2^{(M)})\prod_{l=1}^M m^{(l)}dxdt+\int_Q\phi L^{(\alpha)}dxdt\right]+O\left(\frac{1}{N}\right),\end{aligned}\tag{5.12}$$

where $\xi=N(v_1+\cdots+v_M)$. The right-hand side is $O(1/N)$ since: the F term involves $\prod m^{(l)}\sim e^{-4\pi^2MN^2(T-t)}$, so after dividing by N^M it becomes $O(1/N)$; the $L^{(\alpha)}$ term involves products of at most $M-1$ high-frequency factors and also becomes $O(1/N)$ after division by N^M. The integral on the left concentrates at $t=T$:

$$\int_{T-\tau}^T e^{-4\pi^2MN^2(T-t)}\phi_{-\xi}(x,t)dt\sim\frac{1}{4\pi^2MN^2}\phi_{-\xi}(x,T).$$

Thus, as $N\to\infty$, (5.4) implies that

$$\sum_{|\gamma|=M}(H_1^{(\gamma)}(x)-H_2^{(\gamma)}(x))C_\gamma(v_1,\dots,v_M)=0\quad\text{in the sense of distributions.}$$

By choosing different combinations of unit vectors $v_1,\dots,v_M$, we can isolate each coefficient $H^{(\gamma)}$. For example, to recover $H^{(Me_i)}$, take all $v_l=e_i$; then only $\gamma=Me_i$ contributes and $C_{Me_i}(e_i,\dots,e_i)=(2\pi i)^M\neq0$. More generally, appropriate choices yield linearly independent combinations of all coefficients. Thus $H_1^{(\gamma)}=H_2^{(\gamma)}$ for all $|\gamma|=M$.

Step 5: Recovery of $F^{(M)}$. With $H_1^{(\gamma)}=H_2^{(\gamma)}$ for all $|\gamma|=M$, Equation (5.4) simplifies. Taking $\phi\equiv1$ and $f_l(x)=e^{2\pi i\xi_l\cdot x}$ for various $\xi_l\in\mathbb{Z}^n$, the right-hand side involves only known quantities by the induction hypothesis. By varying $\xi_1,\dots,\xi_M$ over all $\mathbb{Z}^n$ and taking appropriate linear combinations (by varying the constants c_l), we can isolate the Fourier mode $e^{2\pi i\zeta\cdot x}$ for any $\zeta\in\mathbb{Z}^n$. Since $\prod_{l=1}^M m_{\xi_l}$ is bounded away from zero, we conclude $F_1^{(M)}(x)=F_2^{(M)}(x)$. ∎

REMARK 5.1. The recovery of $F^{(M)}$ does not require taking $N\to\infty$; finite but arbitrary frequencies suffice once $H^{(\gamma)}$ have been determined. This reflects the fact that F couples to the measure m which does not have derivative terms that grow with frequency.

5.5. Conclusion of proof.

By induction, we have shown $H_1^{(\beta)}(x)=H_2^{(\beta)}(x)$ for all multi-indices β, and $F_1^{(k)}(x)=F_2^{(k)}(x)$ for all $k\geq1$. Since $H_1,H_2\in\mathcal{H}$ are holomorphic in p, their Taylor series at $p=0$ converge absolutely and uniformly on compact subsets of $\mathbb{C}^n$. Specifically, for any $R>0$,

$$H_j(x,p)=\sum_{k=1}^\infty\sum_{|\beta|=k}H_j^{(\beta)}(x)\frac{p^\beta}{\beta!},$$

with the series converging uniformly for $|p|\le R$. The equality of all Taylor coefficients implies $H_1(x,p)=H_2(x,p)$ for all $x\in\mathbb{T}^n$, $p\in\mathbb{C}^n$, and by restriction, for $p\in\mathbb{R}^n$. Similarly, $F_1,F_2\in\mathcal{F}$ are holomorphic in z, with convergent Taylor series

$$F_j(x,z)=\sum_{k=1}^{\infty}F_j^{(k)}(x)\frac{z^k}{k!},$$

so $F_1(x,z)=F_2(x,z)$ for all $x\in\mathbb{T}^n$, $z\in\mathbb{C}$, hence for $z\in\mathbb{R}$. This completes the proof of Theorem 2.1.

Acknowledgments. The author was partially supported by the Simons Foundation Travel Support for Mathematicians (No. 0007730). The author would like to thank the anonymous reviewers for their insightful comments and remarks, which have greatly improved the manuscript.

REFERENCES

[1] M. Huang, P.E. Caines, and R.P. Malhame, *Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle*, Commun. Inf. Syst., 6(3):221–251, 2006. 1

[2] M. Huang, P.E. Caines, and R.P. Malhame, *Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized ε-Nash equilibrium*, IEEE Trans. Automat. Control, 52(9):1560–1571, 2007. 1

[3] Y.T. Chow, S.W. Fung, S.T. Liu, L. Nurbekyan, and S. Osher, *A numerical algorithm for inverse problem from partial boundary measurement arising from mean field game problem*, Inverse Probl., 39(1):014001, 2023. 1

[4] X. Cao and H. Liu, *Determining a fractional Helmholtz equation with unknown source and scattering potential*, Commun. Math. Sci., 17(7):1861–1876, 2019. 1

[5] X. Cao, Y.-H. Lin, and H. Liu, *Simultaneously recovering potentials and embedded obstacles for anisotropic fractional Schrödinger operators*, Inverse Probl. Imaging, 13(1):197–210, 2019. 1

[6] J. Dieudonn'e, *Foundations of Modern Analysis*, Academic Press, New York, 1960. 3

[7] L. Ding, W. Li, S. Osher, and W. Yin, *A mean field game inverse problem*, J. Sci. Comput., 92:7, 2022. 1

[8] O. Imanuvilov, H. Liu, and M. Yamamoto, *Unique continuation for a mean field game system*, Appl. Math. Lett., 145:108757, 2023. 1

[9] M.V. Klibanov, *A coefficient inverse problem for the mean field games system*, Appl. Math. Optim., 88(2):54, 2023. 1

[10] J.-M. Lasry and P.-L. Lions, *Jeux 'a champ moyen. I. Le cas stationnaire*, C.R. Math. Acad. Sci. Paris, 343(9):619–625, 2006. 1

[11] J.-M. Lasry and P.-L. Lions, *Mean field games*, Jpn. J. Math., 2(1):229–260, 2007. 1

[12] L. Li and W. Wang, *A Calderón type inverse problem for the active scalar equations with fractional dissipation*, Inverse Probl. Imaging, 22:53–64, 2026. 1

[13] B. Liu and W. Wang, *On a partial data inverse problem for the semi-linear wave equation*, arXiv preprint, arXiv:2511.08794, 2025. 1

[14] H. Liu and C.W.K. Lo, *Simultaneously decoding the unknown stationary state and function parameters for mean field games*, arXiv preprint, arXiv:2501.11955, 2025. 1

[15] Y.-H. Lin and H. Liu, *Inverse problems for fractional equations with a minimal number of measurements*, Commun. Anal. Comput., 1(1):72–93, 2023. 1

[16] H. Liu and S. Zhang, *On an inverse boundary problem for mean field games*, arXiv preprint, arXiv:2212.09110, 2022. 1

[17] H. Liu and S. Zhang, *Simultaneously recovering running cost and Hamiltonian in mean field games system*, Commun. Math. Sci., 23(3):813–838, 2025. 1

[18] H. Liu, C.W.K. Lo, and S. Zhang, *Decoding a mean field game by the Cauchy data around its unknown stationary states*, J. Lond. Math. Soc. (2), 111(5):e70173, 2025. 1

[19] M.-H. Ding, H. Liu, and G.-H. Zheng, *Determining a stationary mean field game system from full/partial boundary measurement*, SIAM J. Math. Anal., 57(1):661–681, 2025. 1

[20] M.-H. Ding, H. Liu, and G.-H. Zheng, *Determining internal topological structures and running cost of mean field games with partial boundary measurement*, arXiv preprint, arXiv:2408.08911, 2024. 1

[21] O. Imanuvilov, H. Liu, and M. Yamamoto, *Lipschitz stability for determination of states and inverse source problem for the mean field game equations*, Inverse Probl. Imaging, 18(4):824–859, 2024. 1

[22] M.V. Klibanov, J. Li, and H. Liu, *On the mean field games system with lateral Cauchy data via Carleman estimates*, J. Inverse Ill-Posed Probl., 32(2):277–295, 2024. 1

[23] H. Liu and C.W.K. Lo, *Determining state space anomalies in mean field games*, Nonlinearity, 38(2):025010, 2025. 1

[24] H. Liu, C.W.K. Lo, and S. Zhang, *Inverse problems for mean field games*, arXiv preprint, arXiv:2503.14914, 2025. 1

[25] H. Liu, C. Mou, and S. Zhang, *Inverse problems for mean field games*, Inverse Probl., 39(8):085003, 2023. 1, 2.1, 2.3

[26] Y.-H. Lin and H. Liu, *Inverse Problems for Integro-Differential Operators*, Springer, Cham., 222, 2025. 1

[27] O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Ural'tseva, *Linear and Quasi-linear Equations of Parabolic Type*, Translations of Mathematical Monographs, Amer. Math. Soc., Providence, RI, 23, 1968. 3, 5.1

[28] L. Nirenberg, *Topics in Nonlinear Functional Analysis*, Courant Institute of Mathematical Sciences, New York, 1974. 3